

CUNTZ-LI RELATIONS, INVERSE SEMIGROUPS AND GROUPOIDS

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ABSTRACT. In this paper we show that the universal C^* -algebra satisfying the Cuntz-Li relations is generated by an inverse semigroup of partial isometries. We apply Exel's theory of tight representations to this inverse semigroup. We identify the universal C^* -algebra as the C^* -algebra of the tight groupoid associated to the inverse semigroup.

1. INTRODUCTION

Let R be an integral domain with only finite quotients. Assume that R is not a field and let K be its field of fractions. We denote the set of non-zero elements in R (resp. K) by R^\times (resp. K^\times). In [CL10], Cuntz and Li studied the C^* -algebra, denoted $\mathfrak{A}_r[R]$, on $\ell^2(R)$ generated by the isometries induced by the multiplication and addition operations of the ring R . They showed that it is simple and purely infinite. It was also shown that this C^* -algebra is the universal C^* -algebra generated by isometries satisfying the relations reflecting the semigroup multiplication in $R \rtimes R^\times$ and one more important relation satisfied by the range projections. Also it was shown that $\mathfrak{A}_r[R]$ is Morita-equivalent to a crossed product of the form $C_0(\mathcal{R}) \rtimes (K \rtimes K^\times)$ where \mathcal{R} is a locally compact Hausdorff space. For $R = \mathbb{Z}$, $\mathcal{R} = \mathbb{A}_f$ is the space of finite adeles. Alternate approaches to the algebra $\mathfrak{A}_r[R]$ were considered in [KLQ11], [BE10], and [Sun11].

In [KLQ11], the situation in [CL10] was abstracted. Consider a semidirect product $N \rtimes H$ and a normal subgroup M of N . Let $P := \{a \in H : aMa^{-1} \subset M\}$. Then P is a semigroup. In [KLQ11], under certain hypotheses regarding the pair $(G = N \rtimes H, M)$, the crossed product algebra $C_0(\overline{N}) \rtimes G$ was considered. Here \overline{N} is the profinite completion of N with respect to the group topology induced by the neighbourhood base $\{aMa^{-1}\}_{a \in H}$ at the identity. Let \overline{M} be the closure of M in \overline{N} . In [KLQ11], it was shown that the crossed product algebra $C_0(\overline{N}) \rtimes G$ is Morita-equivalent to the C^* -algebra of the groupoid $\overline{N} \rtimes G|_{\overline{M}}$. In [KLQ11], It was shown that when H is abelian, $C^*(\overline{N} \rtimes G|_{\overline{M}})$

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is the universal C^* -algebra generated by isometries satisfying the relations reflecting the semigroup multiplication in $M \rtimes P$ and one more important relation among the range projections. They also obtained sufficient conditions which will ensure that the reduced C^* -algebra $C_{red}^*(\overline{N} \rtimes G|_{\overline{M}})$ is simple and purely infinite.

Our objective in this paper is to weaken the hypothesis that H is abelian. Instead we assume $H = PP^{-1} = P^{-1}P$. This allows us to consider pairs like $(\mathbb{Q}^n \rtimes GL_n(\mathbb{Q}), \mathbb{Z}^n)$. Also we start with the universal C^* -algebra, denoted $\mathfrak{A}[N \rtimes H, M]$, generated by isometries satisfying the Cuntz-Li relations (See Defn. 2.11.) We show that $\mathfrak{A}[N \rtimes H, M]$ is generated by an inverse semigroup of partial isometries denoted by T . We show that $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to the C^* -algebra of the groupoid \mathcal{G}_{tight} , considered in [Exe08], of the inverse semigroup T . We also identify the groupoid \mathcal{G}_{tight} explicitly and show that \mathcal{G}_{tight} is isomorphic to $\overline{N} \rtimes G|_{\overline{M}}$. The author had done a similar analysis for the Cuntz-Li algebra associated to the ring \mathbb{Z} in [Sun11]. At the end of this paper, we prove a duality result analogous to the duality result obtained in [CL11].

2. SEMIDIRECT PRODUCTS AND THE CUNTZ-LI RELATIONS

Let $G = N \rtimes H$ be a semidirect product and let M be a normal subgroup of N . Let $P := \{a \in H : aMa^{-1} \subset M\}$. Then P is a semigroup containing the identity e . Assume that the following holds.

- (C1) The group $H = PP^{-1} = P^{-1}P$.
- (C2) For every $a \in P$, the subgroup aMa^{-1} is of finite index in M .
- (C3) The intersection $\bigcap_{a \in P} aMa^{-1} = \{e\}$ where e denotes the identity element of G .

Let $\mathcal{U} = \{aMa^{-1} : a \in H\}$. In [KLQ11], the following conditions were required to be satisfied. (Cf. Section 2 in [KLQ11].)

- (E1) Given $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \subset U \cap V$.
- (E2) If $U, V \in \mathcal{U}$ and $U \subset V$ then U is of finite index in V .
- (E3) The intersection $\bigcap_{U \in \mathcal{U}} U = \{e\}$.

We claim that (E1) is equivalent to the condition $H = PP^{-1}$. Assume (E1). Let $a \in H$ be given. Then there exists $c \in H$ such that $a^{-1}Ma \cap M \supseteq cMc^{-1}$. Then $c \in P$ and $ac \in P$. Note that $a = (ac)c^{-1} \in PP^{-1}$. Thus we have $H = PP^{-1}$.

Now suppose $H = PP^{-1}$. First note that for every $a, b \in P$, $aP \cap bP$ is non-empty. Now let $c, d \in H$ be given. Write $c = a_1a_2^{-1}$ and $d = b_1b_2^{-1}$ with $a_i, b_i \in P$. Choose

$\alpha, \beta \in P$ such that $a_1\alpha = b_1\beta$. Let $a := a_1\alpha$. Then $c^{-1}a = a_2\alpha \in P$. Similarly $d^{-1}a \in P$. Hence $aMa^{-1} \subset cMc^{-1} \cap dMd^{-1}$. Thus (E1) holds.

Given (E1), note that (E3) is equivalent to (C3). For if $a \in H$, there exists $b \in P$ such that $aMa^{-1} \cap M \supseteq bMb^{-1}$. Thus for every $a \in H$, $aMa^{-1} \supseteq \bigcap_{b \in P} bMb^{-1}$. Hence $\bigcap_{U \in \mathcal{U}} U = \bigcap_{a \in P} aMa^{-1}$. Thus given (E1), (E3) is equivalent to (C3). Clearly (E2) is equivalent to (C2).

Remark 2.1. In [KLQ11], the Cuntz-Li algebra associated to the pair (Cf. defn 2.11) $(N \rtimes H, M)$ was considered when H is abelian. (Cf. Hypothesis 9.2 and Theorem 9.11 in [KLQ11].) Here, we consider a slightly more general situation. We assume $H = P^{-1}P = PP^{-1}$.

Remark 2.2. The condition $H = P^{-1}P = PP^{-1}$ is equivalent to saying that P generates H and P is right and left reversible i.e. given $a, b \in P$, the intersections $Pa \cap Pb$ and $aP \cap bP$ are non-empty. Cancellative semigroups which are right (or left) reversible are called Ore semigroups. For more details on Ore semigroups, we refer to [CP61].

A semigroup P is called right reversible (left reversible) if $Pa \cap Pb$ (if $aP \cap bP$) is non-empty for every $a, b \in P$.

Throughout this article, whenever we write $G = N \rtimes H$ and M is a normal subgroup of N , we assume that conditions (C1), (C2) and (C3) hold. For $a \in P$, let $M_a = aMa^{-1}$. We will use this notation throughout.

Lemma 2.3. Let $G = N \rtimes H$ and M be a normal subgroup of N . Let $N_0 := \bigcup_{a \in P} a^{-1}Ma$. Then N_0 is a subgroup of N and is invariant under conjugation by H .

Proof. First observe that N_0 is closed under inversion. Let $a, b \in P$ be given. Choose an element c in the intersection $Pa \cap Pb$. Then $a^{-1}Ma \subset c^{-1}Mc$ and $b^{-1}Mb \subset c^{-1}Mc$. Now it follows that N_0 is closed under multiplication. Thus N_0 is a subgroup of N .

Obviously N_0 is invariant under conjugation by P^{-1} . Let $a, b \in P$ be given. Since P is right reversible, there exists $c, d \in P$ such that $ab^{-1} = c^{-1}d$. Now observe that $a(b^{-1}Mb)a^{-1} = c^{-1}(dMd^{-1})c \subset c^{-1}Mc$. Thus it follows that N_0 is closed under conjugation by P . This completes the proof. \square

Remark 2.4. As a consequence of Lemma 2.3, we may very well assume as in [KLQ11] that $N = \bigcup_{a \in P} a^{-1}Ma$.

Let us consider a few examples which fits the setup that we are considering.

Example 2.5 ([CL10]). *Let R be an integral domain such that for every non-zero $m \in R$, the ideal generated by m is of finite index in R . Assume that R is not a field. We denote the field of fractions of R by Q and the set of non-zero elements in Q by Q^\times . The multiplicative group Q^\times acts on Q by multiplication. Now let $N := Q$, $H := Q^\times$ and $M := R$. Then $P = R^\times$ where R^\times denotes the set of non-zero elements in R . Then conditions (C1)-(C3) hold for the pair $(N \rtimes H, M)$.*

Example 2.6 ([KLQ11]). *Let F be a finite group and consider the direct sum $N := \bigoplus_{\mathbb{Z}} F$. Then $H := \mathbb{Z}$ acts on N by shifting. Let $M := \bigoplus_{\mathbb{N}} F$ be the normal subgroup of N . Then it is easily verifiable that the pair $(N \rtimes H, M)$ satisfies the hypothesis (C1)-(C3).*

In the following two examples, we think of elements of \mathbb{Q}^n as column vectors.

Example 2.7. *Let A be a $n \times n$ integer dilation matrix. In other words, A is an $n \times n$ matrix with integer entries such that every complex eigen value of A has absolute value greater than 1. Note that A is invertible over \mathbb{Q} and $|\det(A)| > 1$. The matrix A acts on \mathbb{Q}^n by matrix multiplication and thus induces an action of \mathbb{Z} on \mathbb{Q}^n . We let the generator 1 of \mathbb{Z} act on \mathbb{Q}^n by $1.v = Av$ for $v \in \mathbb{Q}^n$. Let $N := \mathbb{Q}^n$, $H := \mathbb{Z}$ and $M := \mathbb{Z}^n$. Then $P = \mathbb{N}$. Let us verify the hypothesis (C1)-(C3).*

(C1) *Note that H is abelian and $H = PP^{-1} = P^{-1}P$.*

(C2) *For $r \geq 0$, the index of $A^r \mathbb{Z}^n$ is of finite index in \mathbb{Z}^n and in fact its index is $|\det(A)|^r$.*

(C3) *Lemma 4.1 of [EaHR10] implies that the operator norm $\|A^{-m}\|$ converges to 0 as m tends to infinity. Thus if $0 \neq v \in \bigcap_{r=0}^{\infty} A^r \mathbb{Z}^n$, then for every $m \geq 0$, $A^{-m}v \in \mathbb{Z}^n$. Thus we have $1 \leq \|A^{-m}v\| \leq \|A^{-m}\| \|v\|$ which is a contradiction. Thus (C3) holds.*

The case $n = 1$ and $A = p$ where p is a prime number was discussed in [SL10a]. In the previous example, we can consider integer matrices other than dilation matrices. It is possible that (C3) is satisfied for an integer matrix A such that $|\det(A)| > 1$ and $\bigcap_{r>0} A^r \mathbb{Z}^n = \{0\}$ without A being a dilation matrix. In fact we have the following nice characterisation of condition (C3) when $n = 2$.

Lemma 2.8. *Let A be a 2×2 matrix with integer entries. Assume that $|\det(A)| > 1$. Then the following are equivalent.*

- (1) The intersection $\bigcap_{r \geq 0} A^r \mathbb{Z}^2$ is trivial.
 (2) Neither 1 nor -1 is an eigen value of A .

Proof. Suppose $\bigcap_{r \geq 0} A^r \mathbb{Z}^2 = \{0\}$. If 1 is an eigen value of A then there exists a non-zero $v \in \mathbb{Q}^2$ such that $Av = v$. By clearing denominators, we can assume that $v \in \mathbb{Z}^2$. Then clearly $v \in \bigcap_{r \geq 0} A^r \mathbb{Z}^2$. Thus we have shown that 1 is not an eigen value of A . Similarly we can show -1 is not an eigen value of A .

Now assume that neither 1 nor -1 is an eigen value of A . Let $\Gamma_r := A^r \mathbb{Z}^2$ and $\Gamma := \bigcap_{r \geq 0} \Gamma_r$. Since $\Gamma \subset \Gamma_r \subset \mathbb{Z}^2$, we have $[\mathbb{Z}^2 : \Gamma] \geq [\mathbb{Z}^2 : \Gamma_r] = |\det(A)|^r$. Hence Γ cannot be of finite index in \mathbb{Z}^2 . This implies that Γ is of rank at most 1. If Γ is rank 1 then there exists a non-zero $v \in \mathbb{Z}^2$ such that $\Gamma = \mathbb{Z}v$. But $A : \Gamma \rightarrow \Gamma$ is a bijection. Thus it must either be multiplication by 1 or by -1 . In other words, v is an eigen vector for A with eigen value 1 or -1 . This is a contradiction. Thus Γ cannot be of rank 1 which in turn implies $\Gamma = \{0\}$. This completes the proof. \square .

The matrix $A := \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$ has eigen values $\sqrt{3} - 1$ and $-\sqrt{3} - 1$. But A is not a dilation matrix but still (C3) holds for A .

Remark 2.9. *It is not clear to the author whether (C3) can be characterised in terms of eigen values of the matrix in the higher dimensional case.*

Let us now consider an example where H is non-abelian.

Example 2.10. *Let $N = \mathbb{Q}^n$ and H be a subgroup of $GL_n(\mathbb{Q})$ containing the non-zero scalars. Just as in Example 2.7, H acts on N by matrix multiplication. Let $M = \mathbb{Z}^n$. Then P consists of elements of H whose entries are integers.*

- (C1) *Let $A \in H$ be given. Then there exists a non-zero integer m such that $mA = Am \in P$. Hence $H = PP^{-1} = P^{-1}P$.*
 (C2) *For $A \in P$, the subgroup $A\mathbb{Z}^n$ is of finite index and its index is $|\det(A)|$.*
 (C3) *Since $\bigcap_{m \in \mathbb{Z}^\times} m\mathbb{Z}^n = \{0\}$, it follows that $\bigcap_{A \in P} A\mathbb{Z}^n = \{0\}$.*

Definition 2.11. *Let $G := N \rtimes H$ be a semidirect product and M be a normal subgroup of N such that (C1)-(C3) holds. We let $\mathfrak{A}[N \rtimes H, M]$ be the universal C^* -algebra generated by a set of isometries $\{s_a : a \in P\}$ and a set of unitaries $\{u(m) : m \in M\}$ satisfying the*

following relations.

$$\begin{aligned}
s_a s_b &= s_{ab} \\
u(m)u(n) &= u(mn) \\
s_a u(m) &= u(ama^{-1})s_a \\
\sum_{k \in M/M_a} u(k)e_a u(k)^{-1} &= 1
\end{aligned}$$

where e_a denotes the final projection of s_a .

Note that $u(k)e_a u(k)^{-1}$ depends only on the coset $k(M_a)$. Moreover if k_1 and k_2 lie in different cosets of M_a then $u(k_1)e_a u(k_1)^{-1}$ and $u(k_2)e_a u(k_2)^{-1}$ are orthogonal.

For $a \in P$ and $m \in M$, consider the operators S_a and $U(m)$ on $\ell^2(M) \otimes \ell^2(H)$ defined as follows

$$\begin{aligned}
S_a(\delta_n \otimes \delta_b) &:= \delta_{ana^{-1}} \otimes \delta_{ab} \\
U(m)(\delta_n \otimes \delta_b) &:= \delta_{mn} \otimes \delta_b.
\end{aligned}$$

Then $s_a \rightarrow S_a$ and $u(m) \rightarrow U(m)$ gives a representation of $\mathfrak{A}[N \rtimes H, M]$ on the Hilbert space $\ell^2(M) \otimes \ell^2(H)$. Let us call this representation the regular representation and denote its image by $\mathfrak{A}_r[N \rtimes H, M]$.

Remark 2.12. *It should be noted that the regular representation for integral domains considered in [CL10] is different from ours.*

3. AN INVERSE SEMIGROUP FOR THE CUNTZ-LI RELATIONS

The main aim of this section is to show that the C^* -algebra $\mathfrak{A}[N \rtimes H, M]$ is generated by an inverse semigroup of partial isometries. We begin with a lemma similar to Lemma 1 of Section 3.1 in [CL10].

Lemma 3.1. *For every $a, b \in P$, one has*

$$e_a = \sum_{k \in M/M_b} u(aka^{-1})e_{ab}u(aka^{-1})^{-1}$$

Proof. One has

$$\begin{aligned}
e_a &= s_a s_a^* \\
&= s_a \left(\sum_{k \in M/M_b} u(k) e_b u(k)^{-1} \right) s_a^* \\
&= \sum_{k \in M/M_b} u(aka^{-1}) s_a e_b s_a^* u(aka^{-1})^{-1} \\
&= \sum_{k \in M/M_b} u(aka^{-1}) e_{ab} u(aka^{-1})^{-1}
\end{aligned}$$

This completes the proof. \square

Let X be the linear span of $\{u(k)e_b u(k)^{-1} : b \in P, k \in M\}$. Denote the set of projections in X by F . By Lemma 3.1 and the left reversibility of P , it follows that $f \in F$ if and only if there exists $b \in P$ such that f is in the linear span of $\{u(k)e_b u(k)^{-1}\}$. The following lemma is an immediate corollary of Lemma 3.1 and the fact that P is left reversible.

Lemma 3.2. *The set F is a commutative semigroup of projections. Moreover F is invariant under the maps $x \rightarrow s_b x s_b^*$ for every $b \in P$ and $x \rightarrow u(m) x u(m)^{-1}$ for every $m \in M$.*

Now we show that F is also invariant under conjugation by s_a^* for every $a \in P$.

Lemma 3.3. *Let $a \in P$ be given. If $f \in F$, then $s_a^* f s_a \in F$. Moreover, $s_a^* u(m) e_b u(m)^{-1} s_a$ is in the linear span of $\{u(k) e_{a^{-1}c} u(k)^{-1}\}$ where c is any element in $aP \cap bP$.*

Proof. Let $a \in P$ and $f \in F$ be given. First observe that $s_a^* f s_a$ is selfadjoint. Also

$$\begin{aligned}
(s_a^* f s_a)^2 &= s_a^* f s_a s_a^* f s_a \\
&= s_a^* f e_a f s_a \\
&= s_a^* e_a f s_a \quad (\text{Since } F \text{ is commutative}) \\
&= s_a^* f s_a
\end{aligned}$$

Thus $s_a^* f s_a$ is a projection. Now to show that $s_a^* f s_a \in F$, it is enough to consider the case when $f = u(m) e_b u(m)^{-1}$. Now let $c \in aP \cap bP$ and write $c = a\alpha = b\beta$ with $\alpha, \beta \in P$.

Let r_1, r_2, \dots, r_n be distinct representatives of M/M_β . Then by Lemma 3.1, it follows that

$$\begin{aligned} s_a^* u(m) e_b u(m)^{-1} s_a &= \sum_{i=1}^n s_a^* u(mbr_i b^{-1}) e_{b\beta} u(mbr_i b^{-1})^{-1} s_a \\ &= \sum_{i=1}^n s_a^* u(mbr_i b^{-1}) e_{a\alpha} u(mbr_i b^{-1})^{-1} s_a \end{aligned}$$

The term $s_a^* u(mbr_i b^{-1}) e_{a\alpha} u(mbr_i b^{-1})^{-1} s_a$ survives if and only if $e_{a\alpha} u(mbr_i b^{-1}) s_a \neq 0$ and that is if and only if $e_{a\alpha} u(mbr_i b^{-1}) e_a u(mbr_i b^{-1})^{-1} \neq 0$. But by Lemma 3.1 this happens precisely when there exists $t_i \in M/M_\alpha$ such that $mbr_i b^{-1} \equiv at_i a^{-1} \pmod{M_{a\alpha}}$.

Let

$$A := \{i : \text{There exists } t_i \text{ such that } mbr_i b^{-1} \equiv at_i a^{-1} \pmod{M_{a\alpha}}\}.$$

For every $i \in A$, choose t_i such that $mbr_i b^{-1} \equiv at_i a^{-1} \pmod{M_{a\alpha}}$. Now we have

$$\begin{aligned} s_a^* u(m) e_b u(m)^{-1} s_a &= \sum_{i=1}^n s_a^* u(mbr_i b^{-1}) e_{a\alpha} u(mbr_i b^{-1})^{-1} s_a \\ &= \sum_{i \in A} s_a^* u(mbr_i b^{-1}) e_{a\alpha} u(mbr_i b^{-1})^{-1} s_a \\ &= \sum_{i \in A} s_a^* u(at_i a^{-1}) e_{a\alpha} u(at_i a^{-1})^{-1} s_a \\ &= \sum_{i \in A} u(t_i) s_a^* e_{a\alpha} s_a u(t_i)^{-1} \\ &= \sum_{i \in A} u(t_i) e_\alpha u(t_i)^{-1} \end{aligned}$$

This completes the proof. \square

Let us isolate the computation in the previous lemma in a remark. This will be used later.

Remark 3.4. Let $a, b \in P$ be given. Let $c \in aP \cap bP$. Choose α and β in P such that $c = a\alpha = b\beta$. Conjugation by a sends M_α to M_c . Thus we get a map denoted $\pi_\alpha^a : M/M_\alpha \rightarrow M/M_c$. Similarly conjugation by b gives a map $\pi_\beta^b : M/M_\beta \rightarrow M/M_c$. Note that both π_α^a and π_β^b are injective. Denote the quotient map $M \rightarrow M/M_c$ by q_c . For $m \in M$, define

$$A_m := \{r \in M/M_\beta : q_c(m) \pi_\beta^b(r) \in \pi_\alpha^a(M/M_\alpha)\}.$$

Then the computation in Lemma 3.3 can be restated as follows

$$s_a^* u(m) e_b u(m)^{-1} s_a = \sum_{r \in A_m} u \left((\pi_\alpha^a)^{-1} (q_c(m) \pi_\beta^b(r)) \right) e_\alpha u \left((\pi_\alpha^a)^{-1} (q_c(m) \pi_\beta^b(r)) \right)^{-1}.$$

Now we show that $\mathfrak{A}[N \rtimes H, M]$ is generated by an inverse semigroup of partial isometries.

Proposition 3.5. *Let $T := \{s_a^* u(m) f u(m') s_{a'} : m, m' \in M, a, a' \in P, \text{ and } f \in F\}$. Then T is an inverse semigroup of partial isometries containing 0. Moreover the set of projections in T coincides exactly with F . Also the linear span of T is a dense $*$ -subalgebra of $\mathfrak{A}[N \rtimes H, M]$.*

Proof. The fact that T is closed under multiplication follows from the following calculation. Let $a_1, a_2, b_1, b_2 \in P$, $m_1, m_2, n_1, n_2 \in M$ and $e, f \in F$ be given. Choose $c \in P b_1 \cap P a_2$ and write c as $c = \beta b_1 = \alpha a_2$. Observe that

$$\begin{aligned} & s_{a_1}^* u(m_1) e u(m_2) s_{a_2} s_{b_1}^* u(n_1) f u(n_2) s_{b_2} \\ &= s_{a_1}^* u(m_1 m_2) u(m_2^{-1}) e u(m_2) s_\alpha^* s_\alpha s_{a_2} s_{b_1}^* s_\beta^* s_\beta u(n_1) f u(n_1^{-1}) u(n_1 n_2) s_{b_2} \\ &= s_{a_1}^* u(m_1 m_2) u(m_2^{-1}) e u(m_2) s_\alpha^* s_{\alpha a_2} s_{\beta b_1}^* s_\beta u(n_1) f u(n_1^{-1}) u(n_1 n_2) s_{b_2} \\ &= s_{a_1}^* u(m_1 m_2) s_\alpha^* s_\alpha u(m_2^{-1}) e u(m_2) s_\alpha^* s_c s_c^* s_\beta u(n_1) f u(n_1^{-1}) s_\beta^* s_\beta u(n_1 n_2) s_{b_2} \\ &= s_{a_1}^* s_\alpha^* u(\alpha m_1 m_2 \alpha^{-1}) (s_\alpha u(m_2^{-1}) e u(m_2) s_\alpha^*) e_c (s_\beta u(n_1) f u(n_1^{-1}) s_\beta^*) u(\beta n_1 n_2 \beta^{-1}) s_\beta s_{b_2} \\ &= s_{\alpha a_1}^* u(\alpha m_1 m_2 \alpha^{-1}) (s_\alpha u(m_2^{-1}) e u(m_2) s_\alpha^*) e_c (s_\beta u(n_1) f u(n_1^{-1}) s_\beta^*) u(\beta n_1 n_2 \beta^{-1}) s_\beta s_{b_2} \\ &= s_{\alpha a_1}^* u(\alpha m_1 m_2 \alpha^{-1}) (s_\alpha \tilde{e} s_\alpha^*) e_c (s_\beta \tilde{f} s_\beta^*) u(\beta n_1 n_2 \beta^{-1}) s_\beta s_{b_2} \end{aligned}$$

where $\tilde{e} = u(m_2^{-1}) e u(m_2)$ and $\tilde{f} = u(n_1) f u(n_1^{-1})$. The above calculation together with Lemma 3.2 implies that T is closed under multiplication. Obviously T is closed under the involution $*$.

Now let us show that every element of T is a partial isometry. Let $v := s_a^* u(m) f u(m') s_{a'}$ be an element of T . Then

$$v v^* = s_a^* \left(u(m) (f u(m') e_{a'} u(m')^{-1} f) u(m)^{-1} \right) s_a$$

Now Lemma 3.2 and Lemma 3.3 implies that $v v^* \in F$. Thus we have shown that every element of T is a partial isometry and the set of projections in T coincides with F . In other words T is an inverse semigroup.

Since T is closed under multiplication and involution, it follows that the linear span of T is a $*$ -algebra. Moreover T contains $\{s_a : a \in P\}$ and $\{u(m) : m \in M\}$. Thus the linear span of T is dense in $\mathfrak{A}[N \rtimes H, M]$. This completes the proof. \square

The following equality will be used later. Let $a_1, a_2, b_1, b_2 \in P$ and $m_1, m_2 \in M$ be given. Choose $c \in Pb_1 \cap Pa_2$ and write c as $c = \beta b_1 = \alpha a_2$. Now the computation in Proposition 3.5 gives the following equality

$$(3.1) \quad s_{a_1}^* u(m_1) s_{b_1} s_{a_2}^* u(m_2) s_{b_2} = s_{\beta a_1}^* u(\beta m_1 \beta^{-1}) e_c u(\alpha m_2 \alpha^{-1}) s_{\alpha b_2}$$

Remark 3.6. *We also need the following fact. If $v \in T$, let us denote its image in the regular representation by V . Observe that $v \neq 0$ if and only if $V \neq 0$. This is clear for projections in T . Now let $v \in T$ be a non-zero element. Then $vv^* \in F$ is non-zero. Thus $VV^* \neq 0$ which implies $V \neq 0$.*

In the remainder of this article, we reserve the letter T to denote the inverse semigroup in Proposition 3.5 and F to denote the set of projections in T .

4. TIGHT REPRESENTATIONS OF INVERSE SEMIGROUPS

In this section, we show that the identity representation of T in $\mathfrak{A}[N \rtimes H, M]$ is tight in the sense of Exel and the C^* -algebra of the tight groupoid associated to T is isomorphic to $\mathfrak{A}[N \rtimes H, M]$. First let us recall the notion of tight characters and tight representations from [Exe08].

Definition 4.1. *Let S be an inverse semigroup with 0. Denote the set of projections in S by E . A character for E is a map $x : E \rightarrow \{0, 1\}$ such that*

- (1) *the map x is a semigroup homomorphism, and*
- (2) *$x(0) = 0$.*

We denote the set of characters of E by \widehat{E}_0 . We consider \widehat{E}_0 as a locally compact Hausdorff topological space where the topology on \widehat{E}_0 is the subspace topology induced from the product topology on $\{0, 1\}^E$.

For a character x of E , let $A_x := \{e \in E : x(e) = 1\}$. Then A_x is a nonempty set satisfying the following properties.

- (1) The element $0 \notin A_x$.
- (2) If $e \in A_x$ and $f \geq e$ then $f \in A_x$.
- (3) If $e, f \in A_x$ then $ef \in A_x$.

Any nonempty subset A of E for which (1), (2) and (3) are satisfied is called a filter. Moreover if A is a filter then the indicator function 1_A is a character. Thus there is a bijective correspondence between the set of characters and filters. A filter is called an ultrafilter if it is maximal. We also call a character x maximal or an ultrafilter if its

support A_x is maximal. The set of maximal characters is denoted by $\widehat{E_\infty}$ and its closure in $\widehat{E_0}$ is denoted by $\widehat{E_{tight}}$.

We refer to [Sun11] (Corollary 3.3) for the proof of the following lemma.

Lemma 4.2. *Let A be a unital C^* -algebra and $E \subset A$ be an inverse semigroup of projections containing $\{0, 1\}$. Suppose that E contains a finite set $\{e_1, e_2, \dots, e_n\}$ of mutually orthogonal projections such that $\sum_{i=1}^n e_i = 1$. Then for every maximal character x of E , there exists a unique e_i for which $x(e_i) = 1$.*

Let us recall the notion of tight representations of semilattices from [Exe08] and from [Exe09]. The only semilattice we consider is that of an inverse semigroup of projections or in other words the idempotent semilattice of an inverse semigroup. Also our semilattice contains a maximal element 1. First let us recall the notion of a cover from [Exe08].

Definition 4.3. *Let E be an inverse semigroup of projections containing $\{0, 1\}$ and Z be a subset of E . A subset F of Z is called a cover for Z if given a non-zero element $z \in Z$ there exists an $f \in F$ such that $fz \neq 0$. A cover F of Z is called a finite cover if F is finite.*

The following definition is actually Proposition 11.8 in [Exe08]

Definition 4.4. *Let E be an inverse semigroup of projections containing $\{0, 1\}$. A representation $\sigma : E \rightarrow \mathcal{B}$ of the semilattice E in a Boolean algebra \mathcal{B} is said to be tight if $\sigma(0) = 0$ and given $e \neq 0$ in E and for every finite cover F of the interval $[0, e] := \{x \in E : x \leq e\}$, one has $\sup_{f \in F} \sigma(f) = \sigma(e)$.*

Let A be a unital C^* algebra and S be an inverse semigroup containing $\{0, 1\}$. Denote the set of projections in S by E . Let $\sigma : S \rightarrow A$ be a unital representation of S as partial isometries in A . Let $\sigma(C^*(E))$ be the C^* -subalgebra in A generated by $\sigma(E)$. Then $\sigma(C^*(E))$ is a unital, commutative C^* -algebra and hence the set of projections in it is a Boolean algebra which we denote by $\mathcal{B}_{\sigma(C^*(E))}$. We say the representation σ is **tight** if the representation $\sigma : E \rightarrow \mathcal{B}_{\sigma(C^*(E))}$ is **tight**. The proof of the following lemma can be found in [Sun11] (Lemma 3.6, page 7).

Lemma 4.5. *Let X be a compact metric space and $E \subset C(X)$ be an inverse semigroup of projections containing $\{0, 1\}$. Suppose that for every finite set of projections*

$\{f_1, f_2, \dots, f_m\}$ in E , there exists a finite set of mutually orthogonal non-zero projections $\{e_1, e_2, \dots, e_n\}$ in E and a matrix (a_{ij}) such that

$$\sum_{i=1}^n e_i = 1$$

$$f_i = \sum_j a_{ij} e_j.$$

Then the identity representation of E in $C(X)$ is tight.

As in [Sun11], we prove that the identity representation of T in $\mathfrak{A}[N \rtimes H, M]$ is tight.

Proposition 4.6. *The identity representation of T in $\mathfrak{A}[N \rtimes H, M]$ is tight.*

Proof. We apply Lemma 4.5. Let $\{f_1, f_2, \dots, f_n\}$ be a finite set of projections in T . By definition, given i there exists $a_i \in P$ such that f_i is in the linear span of $\{u(k)e_{a_i}u(k)^{-1}\}$. Let $c \in \bigcap_{i=1}^n a_i P$. By Lemma 3.1, it follows that for every i , f_i is in the linear span of $\{u(k)e_c u(k)^{-1} : k \in M/cMc^{-1}\}$. Appealing to Lemma 4.5, we can conclude that the identity representation of T in $\mathfrak{A}[N \rtimes H, M]$ is tight. This completes the proof. \square

Now we show that $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to the C^* -algebra of the groupoid \mathcal{G}_{tight} associated to T . For the convenience of the reader, we recall the construction of the groupoid \mathcal{G}_{tight} , considered in [Exe08], associated to an inverse semigroup with 0.

Let S be an inverse semigroup with 0 and let E denote its set of projections. Note that S acts on $\widehat{E_0}$ partially. For $x \in \widehat{E_0}$ and $s \in S$, define $(x.s)(e) = x(ses^*)$. Then

- The map $x.s$ is a semigroup homomorphism, and
- $(x.s)(0) = 0$.

But $x.s$ is nonzero if and only if $x(ss^*) = 1$. For $s \in S$, define the domain and range of s as

$$D_s := \{x \in \widehat{E_0} : x(ss^*) = 1\}$$

$$R_s := \{x \in \widehat{E_0} : x(s^*s) = 1\}$$

Note that both D_s and R_s are compact and open. Moreover s defines a homeomorphism from D_s to R_s with s^* as its inverse. Also observe that $\widehat{E_{tight}}$ is invariant under the action of S .

Consider the transformation groupoid $\Sigma := \{(x, s) : x \in D_s\}$ with the composition and the inversion being given by:

$$\begin{aligned} (x, s)(y, t) &:= (x, st) \text{ if } y = x.s \\ (x, s)^{-1} &:= (x.s, s^*) \end{aligned}$$

Define an equivalence relation \sim on Σ as $(x, s) \sim (y, t)$ if $x = y$ and if there exists an $e \in E$ such that $x \in D_e$ for which $es = et$. Let $\mathcal{G} = \Sigma / \sim$. Then \mathcal{G} is a groupoid as the product and the inversion respects the equivalence relation \sim . Now we describe a topology on \mathcal{G} which makes \mathcal{G} into a topological groupoid.

For $s \in S$ and U an open subset of D_s , let $\theta(s, U) := \{[x, s] : x \in U\}$. We refer to [Exe08] for the proof of the following proposition. We denote $\theta(s, D_s)$ by θ_s .

Proposition 4.7. *The collection $\{\theta(s, U) : s \in S, U \text{ open in } D_s\}$ forms a basis for a topology on \mathcal{G} . The groupoid \mathcal{G} with this topology is a topological groupoid whose unit space can be identified with $\widehat{E_0}$. Also one has the following.*

- (1) For $s, t \in S$, $\theta_s \theta_t = \theta_{st}$,
- (2) For $s \in S$, $\theta_s^{-1} = \theta_{s^*}$,
- (3) For $s \in S$, θ_s is compact, open and Hausdorff, and
- (4) The set $\{1_{\theta_s} : s \in T\}$ generates the C^* -algebra $C^*(\mathcal{G})$.

We define the groupoid \mathcal{G}_{tight} to be the reduction of the groupoid \mathcal{G} to $\widehat{E_{tight}}$. In [Exe08], it is shown that the representation $s \rightarrow 1_{\theta_s} \in C^*(\mathcal{G}_{tight})$ is tight and any tight representation of S factors through this universal one.

Proposition 4.8. *Let T be the inverse semigroup considered in Proposition 3.5. Denote the tight groupoid associated to T by \mathcal{G}_{tight} . Then $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to $C^*(\mathcal{G}_{tight})$.*

Proof. Let t_a and $v(m)$ be the images of s_a and $u(m)$ in $C^*(\mathcal{G}_{tight})$. By Proposition 4.6 and by the universal property of \mathcal{G}_{tight} , it follows that there exists a homomorphism $\rho : C^*(\mathcal{G}_{tight}) \rightarrow \mathfrak{A}[N \rtimes H, M]$ such that $\rho(t_a) = s_a$ and $\rho(v(m)) = u(m)$.

Given $a \in P$, the projections $\{u(k)e_a u(k)^{-1} : k \in M/M_a\}$ cover the projections in T . Since the representation of T in $C^*(\mathcal{G}_{tight})$ is tight, it follows that

$$\sum_{k \in M/M_a} v(k)(t_a t_a^*) v(k)^{-1} = 1$$

Now the universal property of $\mathfrak{A}[N \rtimes H, M]$ implies that there exists a homomorphism $\sigma : \mathfrak{A}[N \rtimes H, M] \rightarrow C^*(\mathcal{G}_{tight})$ such that $\sigma(s_a) = t_a$ and $\sigma(u(m)) = v(m)$. It is then clear that σ and ρ are inverses of each other. This completes the proof. \square

We identify the groupoid \mathcal{G}_{tight} explicitly in the rest of the article.

5. TIGHT CHARACTERS OF THE INVERSE SEMIGROUP T

In this section, we determine the tight characters of the inverse semigroup T defined in Proposition 3.5. Let

$$\overline{M} := \left\{ (r_a) \in \prod_{a \in P} M/M_a : r_{ab} \equiv r_a \pmod{M_a} \right\}$$

We give \overline{M} the subspace topology induced from the product topology on $\prod_{a \in P} M/M_a$. Here the finite group M/M_a is given the discrete topology. Then \overline{M} is a compact, Hausdorff topological space. Moreover \overline{M} is a topological group. Note that M embeds naturally into \overline{M} via the imbedding $r \rightarrow (r_a := r)$. The map $r \rightarrow (r_a := r)$ is an imbedding since we have assumed that $\bigcap_{a \in P} M_a$ is trivial.

For $b \in P$ and $k \in M$, the set $U_{b,k} := \{(r_a) \in \overline{M} : r_b \equiv k \pmod{M_b}\}$ is an open set. Moreover the collection $\{U_{b,k} : b \in P, k \in M\}$ forms a basis for \overline{M} . If $k \in M$ then clearly $k \in U_{b,k}$ for any $b \in P$. As a consequence, M is dense in \overline{M} .

For $r \in \overline{M}$, let

$$A_r := \{f \in F : f \geq u(r_a)e_a u(r_a)^{-1} \text{ for some } a \in P\}.$$

In the next lemma, we show that for every $r \in \overline{M}$, A_r is an ultrafilter and all ultrafilters are of this form.

Lemma 5.1. *For $r \in \overline{M}$, A_r is an ultrafilter. Moreover any ultrafilter is of the form A_r for some $r \in \overline{M}$.*

Proof: Let $r \in \overline{M}$ be given. First let us show that A_r is a filter. Clearly $0 \notin A_r$. Also if $f_1 \geq f_2$ and $f_2 \in A_r$ then $f_1 \in A_r$. Now suppose that $f_1, f_2 \in A_r$. Then there exists $a_1, a_2 \in P$ such that $f_i \geq u(r_{a_i})e_{a_i}u(r_{a_i})^{-1}$ for $i = 1, 2$. Choose $c \in a_1P \cap a_2P$. Then by Lemma 3.1, it follows that $e_c \leq e_{a_i}$ for $i = 1, 2$. Since $r \in \overline{M}$, it follows that $r_c \equiv r_{a_i} \pmod{M_{a_i}}$ for $i = 1, 2$. Now observe that

$$\begin{aligned} f_1 f_2 &\geq u(r_{a_1})e_{a_1}u(r_{a_1})^{-1}u(r_{a_2})e_{a_2}u(r_{a_2})^{-1} \\ &= u(r_c)e_{a_1}u(r_c)^{-1}u(r_c)e_{a_2}u(r_c)^{-1} \\ &= u(r_c)e_{a_1}e_{a_2}u(r_c)^{-1} \\ &\geq u(r_c)e_c u(r_c)^{-1} \end{aligned}$$

Thus $f_1 f_2 \in A_r$. Thus we have shown that A_r is a filter.

Now we show A_r is maximal. Let A be a filter which contains A_r . Consider an element $f \in A$. By definition there exists $a \in P$ and scalars $\alpha_k \in \{0, 1\}$ such that

$$f = \sum_{k \in M/M_a} \alpha_k u(k) e_a u(k)^{-1}.$$

But both f and $u(r_a) e_a u(r_a)^{-1}$ belong to A and hence their product belongs to A . Thus the product $f u(r_a) e_a u(r_a)^{-1}$ is non-zero. This implies that $\alpha_{r_a} = 1$. Thus we have $f \geq u(r_a) e_a u(r_a)^{-1}$ or in other words $f \in A_r$. Hence $A = A_r$. This proves that A_r is maximal.

Let A be an ultrafilter. By Lemma 4.2, it follows that for every $a \in P$, there exists a unique $r_a \in M/M_a$ such that $u(r_a) e_a u(r_a)^{-1} \in A$. Let $r := (r_a)$. We claim that $r \in \overline{M}$. Let $a, b \in P$ be given. By Lemma 3.1, we have

$$(5.2) \quad u(r_a) e_a u(r_a)^{-1} = \sum_{k \in M/M_b} u(r_a a k a^{-1}) e_{ab} u(r_a k a k^{-1})^{-1}$$

Since A is a filter containing $u(r_a) e_a u(r_a)^{-1}$ and $u(r_{ab}) e_{ab} u(r_{ab})^{-1}$, it follows that their product is non-zero. This fact together with Equation 5.2 implies that there exists $k \in M$, such that $r_{ab} \equiv r_a (a k a^{-1}) \pmod{M_{ab}}$. Thus $r_{ab} \equiv r_a \pmod{M_a}$ for every $a, b \in P$. As a result, we have $r \in \overline{M}$. Since A is a filter it follows that $A_r \subset A$. We have already proved that A_r is maximal. Thus $A = A_r$. This completes the proof. \square

The following proposition identifies the tight characters of T .

Proposition 5.2. *The map $\overline{M} : r \rightarrow A_r \in \widehat{F_{tight}}$ is a homeomorphism.*

Proof. It is clear from the definition that $r \rightarrow A_r$ is one-one. Let us denote this map by ϕ . We show ϕ is continuous. Consider a net r^α in \overline{M} converging to r . We denote the indicator function of a set A by 1_A . Let $f \in F$ be given. Then there exists $a \in P$ and scalars α_k such that

$$f = \sum_k \alpha_k u(k) e_a u(k)^{-1}$$

Then we have

$$1_{A_r^\alpha}(f) = \sum_k \alpha_k \delta_{r_a^\alpha, k}$$

Since $r_a^\alpha = r_a$ eventually, it follows that $1_{A_r^\alpha}(f)$ converges to $1_{A_r}(f)$. This shows that $r \rightarrow A_r$ is continuous.

Now Lemma 5.1 implies that ϕ has range $\widehat{F_\infty}$. Since \overline{M} is compact, it follows that $\widehat{F_\infty}$ is compact and hence closed. Thus $\widehat{F_\infty} = \widehat{F_{tight}}$. Thus $\phi : \overline{M} \rightarrow \widehat{F_\infty}$ is one-one, onto

and continuous. Since \overline{M} is compact, it follows that ϕ is in fact a homeomorphism. This completes the proof. \square

From now on we will simply denote A_r by r and $1_{A_r}(f)$ by $r(f)$.

6. THE GROUPOID \mathcal{G}_{tight} OF THE INVERSE SEMIGROUP T

In this section, we will identify the tight groupoid \mathcal{G}_{tight} associated to the inverse semigroup. Throughout this section, we assume $N = \bigcup_{a \in P} a^{-1}Ma$. By Remark 2.4, we can very well assume this. There is another natural groupoid which arises out of the following construction.

For every $a \in P$, the co-isometry s_a^* will give rise to an injection on \overline{M} and the unitary $u(m)$ for $m \in M$ will act as a bijection on \overline{M} . Thus we get an action of the semigroup $M \rtimes P$, as injections, on \overline{M} . Now the space \overline{M} can be enlarged to a space \overline{N} and the action of $M \rtimes P$ can be dilated to get an action of $G = N \rtimes H$ on \overline{N} . We can then consider the transformation groupoid $\overline{N} \rtimes G$. But the unit space of \mathcal{G}_{tight} is \overline{M} . Thus we restrict the transformation groupoid $\overline{N} \rtimes G$ to \overline{M} and prove that it is isomorphic to \mathcal{G}_{tight} .

This dilation procedure has appeared in several works [See [Lac00], [SL10b]]. The basic principle goes back to [Ore31].

First let us explain the action of $M \rtimes P$ on \overline{M} . The action of M on \overline{M} is by left multiplication as M is a subgroup of \overline{M} . Let $a \in P$ and $r \in \overline{M}$ be given. For $b \in P$, choose $c \in aP \cap bP$ and write c as $c = a\alpha = b\beta$. We will use the notation as in Remark 3.4. Note that $M_c \subset M_b$ and we denote the induced quotient map $M/M_c \rightarrow M/M_b$ by $q_{b,c}$. Define $m_b = q_{b,c}(\pi_\alpha^a(r_\alpha))$. First let us show that m_b depends only on a and b and not on the choices made.

Suppose $c_1 = a\alpha_1 = b\beta_1$ and $c_2 = a\alpha_2 = b\beta_2$. Choose $\gamma_1, \gamma_2 \in P$ such that $\alpha_1\gamma_1 = \alpha_2\gamma_2$. Note that this implies $c_1\gamma_1 = c_2\gamma_2$. Now we have

$$\begin{aligned} q_{b,c_i}\pi_{\alpha_i}^a(r_{\alpha_i}) &= q_{b,c_i}\left(\pi_{\alpha_i}^a(q_{\alpha_i,\alpha_i\gamma_i}(r_{\alpha_i\gamma_i}))\right) \\ &= q_{b,c_i}\left(q_{c_i,c_i\gamma_i}(\pi_{\alpha_i\gamma_i}^a(r_{\alpha_i\gamma_i}))\right) \\ &= q_{b,c_i\gamma_i}(\pi_{\alpha_i\gamma_i}^a(r_{\alpha_i\gamma_i})). \end{aligned}$$

Note that the right hand side is constant for $i = 1, 2$. Thus we have

$$q_{b,c_1}(\pi_{\alpha_1}^a(r_{\alpha_1})) = q_{b,c_2}(\pi_{\alpha_2}^a(r_{\alpha_2})).$$

This shows that m_b is well defined. We leave it to the reader to check that $\tilde{m} = (m_b) \in \overline{M}$.

On M , the action of P is the usual conjugation. From now on, we denote the element \tilde{m} by ara^{-1} . This way P acts on \overline{M} injectively and continuously. This action of P together with the left multiplication action of M defines an action of $M \rtimes P$ on \overline{M} (as injective, continuous transformations). We leave the details to the reader.

Lemma 6.1. *For $a \in P$, the kernel of the projection map $\overline{M} \ni (y_b) \rightarrow y_a \in M/M_a$ is $a\overline{M}a^{-1}$.*

Proof. By definition, it follows that $a\overline{M}a^{-1}$ is in the kernel of the a^{th} projection. Now let $y = (y_b)$ be such that $y_a = 1$. Since M is dense in \overline{M} , there exists a sequence $y^n \in M$ such that $y^n \rightarrow y$ in \overline{M} . As M/M_a is finite, we can without loss of generality assume that $y^n \in M_a$ for every n . Thus there exists $x^n \in M$ such that $y^n = ax^n a^{-1}$. But \overline{M} is compact. Thus, by passing to a subsequence if necessary, we can assume that x^n converges to an element say $x \in \overline{M}$. Since conjugation by a is continuous, it follows that $y^n = ax^n a^{-1}$ converges to axa^{-1} . But y^n converges to y . Thus $axa^{-1} = y$. This completes the proof. \square

Now let us explain the dilation procedure that we promised at the beginning of this section. Consider the set $\overline{M} \times P$ and define a relation on $\overline{M} \times P$ by $(x, a) \sim (y, b)$ if there exists $\alpha, \beta \in P$ such that $\alpha a = \beta b$ and $\alpha x \alpha^{-1} = \beta y \beta^{-1}$. We leave the following routine checking to the reader.

- (1) The relation \sim is an equivalence relation. We denote the equivalence class containing (x, a) by $[(x, a)]$.
- (2) Let $\overline{N} := \overline{M} \times P / \sim$. Then \overline{N} is a group. The multiplication on \overline{N} is defined as follows. For $a, b \in P$, choose α and β such that $\alpha a = \beta b$. Then

$$[(x, a)][(y, b)] = [(\alpha x \alpha^{-1} \beta y \beta^{-1}, \alpha a)]$$

The identity element of \overline{N} is $[(e, e)]$ where (e, e) is the identity element of $\overline{M} \times P$ and the inverse of $[(x, a)]$ is $[(x^{-1}, a)]$.

- (3) The group \overline{N} is a locally compact Hausdorff topological group when \overline{N} is given the quotient topology. Here P is given the discrete topology.
- (4) The map $M \ni x \rightarrow [(x, e)] \in \overline{N}$ is a topological embedding. Thus \overline{M} can be viewed as a subset of \overline{N} . Moreover \overline{M} is a compact open subgroup of \overline{N} .
- (5) The map $N \ni a^{-1}ma \rightarrow [(m, a)] \in \overline{N}$ is an embedding. When N is viewed as a subset of \overline{N} via this embedding, N is dense in \overline{N} . Also $N \cap \overline{M} = M$.
- (6) Let $a \in P$ be given. Define a map $\phi_a : \overline{N} \rightarrow \overline{N}$ as follows. Given $[(x, b)] \in \overline{N}$, choose $\alpha, \beta \in P$ such that $\alpha a = \beta b$. Define $\phi_a([(x, b)]) = [\beta x \beta^{-1}, \alpha]$. One

checks that ϕ_a is well defined. Moreover for $a \in P$, ϕ_a is a homeomorphism with ϕ_a^{-1} given by $\phi_a^{-1}[(x, b)] = [(x, ba)]$. Note that ϕ_a restricted to N is the usual conjugation. Also $\phi_a \phi_b = \phi_{ab}$ for $a, b \in P$. For $m \in M$, define $\psi_m : \overline{N} \rightarrow \overline{N}$ as $\psi_m([(x, a)]) = [(ama^{-1}x, a)]$. That is ψ_m is just left multiplication by m . One also has the following commutation relation. For $a \in P$ and $m \in M$,

$$\phi_a \psi_m = \psi_{ama^{-1}} \phi_a.$$

- (7) Since we have assumed that $N = \bigcup_{a \in P} a^{-1}Ma$, it follows that any element of $g \in G = N \rtimes H$ can be written as $g = a^{-1}mb$ with $a, b \in P$ and $m \in M$. The map $a^{-1}mb \rightarrow \phi_a^{-1} \psi_m \phi_b$ is well defined and defines an action of G on \overline{N} . If $h = a^{-1}b \in H$ and $x \in \overline{N}$, we denote $\phi_a^{-1} \phi_b(x)$ as $h x h^{-1}$. If $n = a^{-1}ma$ and $x \in \overline{N}$, we denote $\phi_a^{-1} \psi_m \phi_a(x)$ as $n x$.
- (8) Note that $\overline{N} = \bigcup_{a \in P} a^{-1} \overline{M} a$.
- (9) **Universal Property:** Let L be a locally compact Hausdorff topological group on which H acts by group homomorphism. Suppose that K is a compact open subgroup of L which is invariant under P and $L = \bigcup_{a \in P} a^{-1}K$. If $\phi : \overline{M} \rightarrow K$ is a P -equivariant continuous bijection then the map $\overline{N} \ni a^{-1}xa \rightarrow a^{-1} \cdot \phi(x) \in L$ is a topological isomorphism and is H -equivariant.

Remark 6.2. *It is not difficult to show by using (9) that \overline{N} is the pro-finite completion of N when N is given the topology induced by the neighbourhood base $\{aMa^{-1} : a \in H\}$ at the identity. In [KLQ11], the pro-finite completion model of \overline{N} is used.*

When considering transformation groupoids, we consider only right actions of groups and thus we change the above left action of G on \overline{N} to a right action simply by defining $x.g = g^{-1}x$ for $x \in \overline{N}$ and $g \in G$. Now consider the transformation groupoid $\overline{N} \rtimes G$ and restrict it to \overline{M} . We show that the groupoid \mathcal{G}_{tight} of the inverse semigroup T is isomorphic to the groupoid $\overline{N} \rtimes G|_{\overline{M}}$ i.e. to the transformation groupoid $\overline{N} \rtimes G$ restricted to the unit space \overline{M} . We will start with two lemmas which will be extremely useful to prove this.

Lemma 6.3. *If $a_1^{-1}m_1b_1 = a_2^{-1}m_2b_2$ then $s_{a_1}^* u(m_1) s_{b_1} = s_{a_2}^* u(m_2) s_{b_2}$.*

Proof. Suppose $a_1^{-1}m_1b_1 = a_2^{-1}m_2b_2$. Then $a_1^{-1}m_1a_1 = a_2^{-1}m_2a_2$ and $a_1^{-1}b_1 = a_2^{-1}b_2$. Choose $\beta_1, \beta_2 \in P$ such that $\beta_1b_1 = \beta_2b_2$. Then $a_1a_2^{-1} = \beta_1^{-1}\beta_2 = b_1b_2^{-1}$. Hence

$\beta_1 m_1 \beta_1^{-1} = \beta_2 m_2 \beta_2^{-1}$. Now observe that

$$\begin{aligned}
 s_{a_1}^* u(m_1) s_{b_1} &= s_{a_1}^* u(m_1) s_{\beta_1}^* s_{\beta_1} s_{b_1} \\
 &= s_{a_1}^* s_{\beta_1}^* u(\beta_1 m_1 \beta_1^{-1}) s_{\beta_1} s_{b_1} \\
 &= s_{\beta_1 a_1}^* u(\beta_1 m_1 \beta_1^{-1}) s_{\beta_1} s_{b_1} \\
 &= s_{\beta_2 a_2}^* u(\beta_2 m_2 \beta_2^{-1}) s_{\beta_2} s_{b_2} \\
 &= s_{a_2}^* s_{\beta_2}^* u(\beta_2 m_2 \beta_2^{-1}) s_{\beta_2} s_{b_2} \\
 &= s_{a_2}^* u(m_2) s_{\beta_2}^* s_{\beta_2} s_{b_2} \\
 &= s_{a_2}^* u(m_2) s_{b_2}
 \end{aligned}$$

This completes the proof. \square

Lemma 6.4. *In \mathcal{G}_{tight} , $[(r, s_a^* u(m) f u(n) s_b)] = [(r, s_a^* u(mn) s_b)]$.*

Proof. First observe that $[(r, s_a^*)][r, s_a^* u(m) f u(n) s_b] = [(r, s_a^* u(m) f u(n) s_b)]$. Thus it is enough to consider the case when a is the identity element of P . Now let $s = u(m) f u(n) s_b$, $t = u(mn) s_b$ and $e = u(m) f u(m)^{-1}$. Observe that $s = et$. Thus $ss^* = ett^*e$. Hence $r(ss^*) = 1$ implies $r(e) = 1$ and $r(tt^*) = 1$. Moreover $es = s = et$. Thus $[(r, s)] = [(r, t)]$. This completes the proof. \square

Now we can state our main theorem.

Theorem 6.5. *Let $\phi : \overline{N} \rtimes G|_{\overline{M}} \rightarrow \mathcal{G}_{tight}$ be the map defined by*

$$\phi((x, a^{-1}mb)) = [(x, s_a^* u(m) s_b)].$$

Then ϕ is a topological groupoid isomorphism.

Proof. First let us show that ϕ is well defined. Let $(x, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$. Then by definition, there exists $y \in \overline{M}$ such that $m^{-1}axa^{-1} = byb^{-1}$. Choose α and β in P such that $c := a\alpha = b\beta$. By definition, this means that $\pi_\alpha^a(x_\alpha) \equiv q_c(m)\pi_\beta^b(y_\beta)$. Now Remark 3.4 implies that

$$s_a^* u(m) e_b u(m)^{-1} s_a \geq u(x_\alpha) e_\alpha u(x_\alpha)^{-1}.$$

Hence $x(s_a^* u(m) e_b u(m)^{-1} s_a) = 1$. Thus we have shown that ϕ is well-defined.

Before we show ϕ is a surjection, let us show that if $[(x, s_a^* u(m) s_b)] \in \mathcal{G}_{tight}$ then $(x, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$. To that effect, assume that $x(s_a^* u(m) e_b u(m)^{-1} s_a) = 1$. Choose $c \in aP \cap bP$ and write $c = a\alpha = b\beta$. By Remark 3.4, it follows that there exists $y \in M/M_\beta$ such that $q_c(m^{-1})\pi_\alpha^a(x_\alpha) = \pi_\beta^b(y)$. This implies that the b^{th} co-ordinate of

$m^{-1}axa^{-1}$ is 1 i.e. the identity element of M/M_b . Now Lemma 6.1 implies that there exists $z \in \overline{M}$ such that $m^{-1}axa^{-1} = bzb^{-1}$. Hence $(x, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$. Surjectivity is then an immediate consequence of Lemma 6.4.

Now we show ϕ is injective. Suppose $[(x, s_{a_1}^* u(m_1)s_{b_1})] = [(x, s_{a_2}^* u(m_2)s_{b_2})]$. Then there exists a projection $e \in F$ such that $0 \neq e(s_{a_1}^* u(m_1)s_{b_1}) = e(s_{a_2}^* u(m_2)s_{b_2})$. We can without loss of generality assume that $e = u(r_c)e_c u(r_c)^{-1}$. By Remark 3.6 and by reading the above equality in the regular representation, we immediately obtain $a_1^{-1}b_1 = a_2^{-1}b_2$ and $a_1^{-1}m_1b_1 = a_2^{-1}m_2b_2$. This implies that ϕ is injective.

Now let us show that ϕ is a groupoid morphism. First we show that ϕ preserves the range and source. By definition, ϕ preserves the range. Observe that ϕ is continuous and this is a direct consequence of Proposition 5.2. Let $\gamma = (x, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$. Since M is dense in \overline{M} there exists a sequence $x_n \in M$ such that x_n converges to x . Moreover the action of G on \overline{N} is continuous and \overline{M} is compact and open. Thus we can assume that $(x_n, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$ for every n . By definition, there exists $y \in \overline{M}$ such that $axa^{-1} = mbyb^{-1}$. Also let y_n be such that $ax_n a^{-1} = mby_n b^{-1}$.

To keep things clear, if $z \in \overline{M}$, we denote the character determined by z as ξ_z . Let $v := s_a^* u(m)s_b$. Now if can show that $\xi_{x_n} \cdot v = \xi_{y_n}$ then it will follow from continuity of ϕ that $\xi_x \cdot v = \xi_y$. Thus we only need to show that $s(\phi(\gamma)) = \phi(s(\gamma))$ for $\gamma = (x, a^{-1}mb)$ with $x \in M$.

Now let $(x, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$ with $x \in M$. Then there exists $y \in M$ such that $axa^{-1} = mbyb^{-1}$. Let $v = s_a^* u(m)s_b$. To show $\xi_x \cdot v = \xi_y$, as ξ_y is maximal, it is enough to show that the support of ξ_y is contained in $\xi_x \cdot v$. Again it is enough to show that $u(y)e_c u(y)^{-1}$ is in the support of $\xi_x \cdot v$. Choose α, β such that $a\alpha = bc\beta$. Note that

$$\begin{aligned}
vu(y)e_c u(y)^{-1}v^* &= s_a^* u(m)s_b u(y)e_c u(y)^{-1}s_b^* u(m)^{-1}s_a \\
&= s_a^* u(mbyb^{-1})s_b e_c s_b^* u(mbyb^{-1})^{-1}s_a \\
&= s_a^* u(axa^{-1})e_{bc} u(axa^{-1})^{-1}s_a \\
&= u(x)s_a^* e_{bc}s_a u(x)^{-1} \\
&\geq u(x)s_a^* e_{bc\beta}s_a u(x)^{-1} \\
&= u(x)s_a^* e_{a\alpha}s_a u(x)^{-1} \\
&= u(x)e_\alpha u(x)^{-1} \in \text{supp}(\xi_x)
\end{aligned}$$

Hence $u(y)e_c u(y)^{-1}$ is in the support of $\xi_x \cdot v$. Thus we have shown that $\xi_x \cdot v = \xi_y$. This proves that ϕ preserves the source.

Now we show ϕ preserves multiplication. Let $\gamma_1 = (x_1, a_1^{-1}m_1b_1)$ and $\gamma_2 = (x_2, a_2^{-1}m_2b_2)$. Since ϕ preserves the range and source, it follows that γ_1 and γ_2 are composable if and only if $\phi(\gamma_1)$ and $\phi(\gamma_2)$ are composable. Choose $\alpha, \beta \in P$ such that $\beta b_1 = \alpha a_2$. . Now

$$\begin{aligned} \phi(\gamma_1)\phi(\gamma_2) &= [(x_1, s_{a_1}^* u(m_1) s_{b_1} s_{a_2}^* u(m_2) s_{b_2})] \\ &= [(x_1, s_{\beta a_1}^* u(\beta m_1 \beta^{-1}) e_{\alpha a_2} u(\alpha m_2 \alpha^{-1}) s_{\alpha b_2})] \quad (\text{by Eq. 3.1}) \\ &= [(x_1, s_{\beta a_1}^* u(\beta m_1 \beta^{-1} \alpha m_2 \alpha^{-1}) s_{\alpha b_2})] \quad (\text{by Remark 6.4}) \\ &= \phi(\gamma_1 \gamma_2). \end{aligned}$$

It is easily verifiable that ϕ preserves inversion.

For an open subset U of \overline{M} and $g = a^{-1}mb$, consider the open set

$$\theta(U, g) := \{x \in \overline{M} : x.g \in \overline{M}\}$$

The collection $\{\theta(U, g)\}$ forms a basis for $\overline{N} \rtimes G|_{\overline{M}}$. Moreover $\phi(\theta(U, g)) = \theta(U, s_a^* u(m) s_b)$. Thus ϕ is an open map. Thus we have shown that ϕ is a homeomorphism. This completes the proof. \square

Corollary 6.6. *The algebra $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to $C^*(\overline{N} \rtimes G|_{\overline{M}})$.*

Proof. This follows from Theorem 6.5 and Proposition 4.8. \square

7. SIMPLICITY OF $\mathfrak{A}_r[N \rtimes H, M]$

Let us recall a few definitions from [AD97]. Let \mathcal{G} be an r -discrete groupoid and we denote its unit space by \mathcal{G}^0 . The relation \sim defined by $x \sim y$ if and only if there exists $\gamma \in \mathcal{G}$ such that $s(\gamma) = x$ and $r(\gamma) = y$ is an equivalence relation on \mathcal{G}^0 . A subset $E \subset \mathcal{G}^0$ is said to be invariant if given $x \in E$ and $y \sim x$ then $y \in E$. For $x \in \mathcal{G}$, let $\mathcal{G}(x) := \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma) = x\}$ be the isotropy group of x .

A subset $S \subset \mathcal{G}$ is said to be a bi-section if the range and source maps restricted to S are one-one. If S is a bisection, let $\alpha_S : r(S) \rightarrow s(S)$ be defined by $\alpha_S := s \circ r^{-1}$.

The groupoid \mathcal{G} is said to be

- minimal if the only non-empty, open invariant subset of \mathcal{G}^0 is \mathcal{G}^0 .
- topologically principal if the set of $x \in \mathcal{G}^0$ for which $\mathcal{G}(x) = \{x\}$ is dense in \mathcal{G}^0 .
- locally contractive if for every non-empty open subset U of \mathcal{G}^0 , there exists an open subset $V \subset U$ and an open bisection S with $\overline{V} \subset s(S)$ and $\alpha_{S^{-1}}(\overline{V})$ not contained in V .

Conjugation by P on M gives rise to a semigroup homomorphism from P to the semigroup of injective maps on M . In [KLQ11], the action of P on M is called an effective action if the above semigroup homomorphism is injective i.e. given $h \in H$ with $h \neq 1$, then there exists $s \in M$ such that $hsh^{-1} \neq s$. In [KLQ11], the following facts were proved about the transformation groupoid $\overline{N} \rtimes G$.

- (1) The groupoid $\overline{N} \rtimes G$ is minimal and locally contractive.
- (2) The groupoid $\overline{N} \rtimes G$ is topologically principal if and only if P acts effectively on M .
- (3) Thus the reduced C^* -algebra $C_{red}^*(\overline{N} \rtimes G)$ is simple and purely infinite if P acts effectively on M . [Refer to [AD97]].

Analogous statements hold for the groupoid \mathcal{G}_{tight} associated to the inverse semigroup T .

Remark 7.1. In [KLQ11], only the if part (in (2)) was proved. But then the other direction i.e. if $\overline{N} \rtimes G$ is topologically principal then P acts effectively on M is easy to verify.

Also note that \overline{M} is a closed subset of \overline{N} which meets each G orbit of \overline{N} . Moreover \overline{M} is open as well. Hence by appealing to Example 2.7 in [MRW87], we conclude that $C^*(\overline{N} \rtimes G)$ and $C^*(\overline{N} \rtimes G|_{\overline{M}})$ are Morita-equivalent.

We end this section by showing that $\mathfrak{A}_r[N \rtimes H, M]$ is isomorphic to the reduced C^* -algebra $C_{red}^*(\mathcal{G}_{tight})$.

Proposition 7.2. Let $\mathcal{G} := \overline{N} \rtimes G|_{\overline{M}}$. Then the reduced C^* -algebra of the groupoid \mathcal{G} is isomorphic to $\mathfrak{A}_r[N \rtimes H, M]$.

Proof. Let e be the identity element of \overline{M} . Define $\mathcal{G}^e := \{\gamma \in \mathcal{G} : r(\gamma) = e\}$. Then $\mathcal{G}^e := \{(e, hm) : m \in M, h \in H\}$. Thus $L^2(\mathcal{G}^e)$ can be identified with $\ell^2(M) \otimes \ell^2(H)$. Consider the representation π_e of $C_{red}^*(\mathcal{G})$ on $L^2(\mathcal{G}^e)$ defined as follows. For $f \in C_c(\mathcal{G})$, define $\pi_e(f)$ by the following formula.

$$(\pi_e(f)(\xi))(\gamma) := \sum_{\gamma_1 \in \mathcal{G}^e} f(\gamma^{-1}\gamma_1)\xi(\gamma_1)$$

Since M is dense in \overline{M} , it follows that the largest open invariant set not containing e is the empty set. Hence π_e is faithful.

For $a \in P$ and $m \in M$, we let S_a and $U(m)$ be the images of s_a and $u(m)$ in $C_{red}^*(\mathcal{G})$. Let $\{\delta_m \otimes \delta_b : m \in M, b \in H\}$ be the canonical basis of $\ell^2(M) \otimes \ell^2(H)$. Consider the

unitary operator V on $\ell^2(M) \otimes \ell^2(H)$ defined by

$$V(\delta_m \otimes \delta_b) := \delta_{m^{-1}} \otimes \delta_{b^{-1}}$$

For $a \in P$ and $k \in M$, we leave it to the reader to check the following equality.

$$\begin{aligned} V\pi_e(S_a)V^*(\delta_m \otimes \delta_b) &= \delta_{ama^{-1}} \otimes \delta_{ab} \\ V\pi_e(U(k))V^*(\delta_m \otimes \delta_b) &= \delta_{km} \otimes \delta_b \end{aligned}$$

Since $\{S_a : a \in P\}$ and $\{U(k) : k \in M\}$ generate $C_{red}^*(\mathcal{G})$, it follows that $C_{red}^*(\mathcal{G})$ is isomorphic to $\mathfrak{A}_r[N \rtimes H, M]$. This completes the proof. \square

We now show that Corollary 6.6 and Proposition 7.2 can also be expressed in terms of crossed products as in [KLQ11]. We need to digress a bit before we do this.

Let \mathcal{G} be an r -discrete, locally compact and Hausdorff groupoid. Let $Y \subset \mathcal{G}^0$ be a compact open subset of the unit space. Assume that Y meets each orbit of \mathcal{G}^0 . Let

$$\begin{aligned} \mathcal{G}^Y &:= \{\gamma \in \mathcal{G} : s(\gamma) \in Y\} \\ \mathcal{G}_Y^Y &:= \{\gamma \in \mathcal{G} : s(\gamma), r(\gamma) \in Y\} \end{aligned}$$

Since Y is clopen, it follows that \mathcal{G}^Y and \mathcal{G}_Y^Y are clopen. Thus if $f \in C_c(\mathcal{G}^Y)$, then f can be extended to an element in $C_c(\mathcal{G})$ by declaring its value to be zero outside \mathcal{G}^Y . Thus we have the inclusion $C_c(\mathcal{G}^Y) \subset C_c(\mathcal{G})$. Similarly, we have the inclusion $C_c(\mathcal{G}_Y^Y) \subset C_c(\mathcal{G}^Y)$. The algebra $C_c(\mathcal{G}_Y^Y)$ is a $*$ -subalgebra of $C_c(\mathcal{G})$.

The space $C_c(\mathcal{G}^Y)$ is a pre-Hilbert $C_c(\mathcal{G}_Y^Y) \subset C^*(\mathcal{G}_Y^Y)$ module with the inner product and the right multiplication given by

$$\begin{aligned} \langle f_1, f_2 \rangle(\gamma) &= \sum_{\gamma_1 \gamma_2 = \gamma} \overline{f_1(\gamma_1^{-1})} f_2(\gamma_2) \text{ for } \gamma \in \mathcal{G}_Y^Y, f_1, f_2 \in C_c(\mathcal{G}^Y) \\ (f \cdot g)(\gamma) &= \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) \text{ for } \gamma \in \mathcal{G}^Y, f \in C_c(\mathcal{G}^Y), g \in C_c(\mathcal{G}_Y^Y) \end{aligned}$$

Moreover there is left action of $C_c(\mathcal{G})$ on $C_c(\mathcal{G}^Y)$ and it is given by

$$\begin{aligned} (f \cdot \phi)(\gamma) &= (f * \phi)(\gamma) \\ &= \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) \phi(\gamma_2) \end{aligned}$$

for $\gamma \in \mathcal{G}^Y$, $f \in C_c(\mathcal{G})$ and $\phi \in C_c(\mathcal{G}^Y)$.

Now Theorem 2.8 and Example 2.7 of [MRW87] implies the following. The “completion” of $C_c(\mathcal{G})$ - $C_c(\mathcal{G}_Y^Y)$ bimodule $C_c(\mathcal{G}^Y)$ is a $C^*(\mathcal{G})$ - $C^*(\mathcal{G}_Y^Y)$ imprimitivity bimodule implementing a strong Morita equivalence between $C^*(\mathcal{G})$ and $C^*(\mathcal{G}_Y^Y)$.

Let us denote the completion of $C_c(\mathcal{G}^Y)$ by \mathcal{E} . For $x, y \in \mathcal{E}$, let $\theta_{x,y}$ be the compact operator on \mathcal{E} defined by $\theta_{x,y}(z) = x \langle y, z \rangle$. For $x \in \mathcal{E}$, the operator norm of $\theta_{x,x}$ is $\|x\|^2$.

The following proposition has also appeared in [Li12]. (See Lemma 5.18 in [Li12].) The proof is exactly as in [Li12]. We include the proof for the sake of completeness.

Proposition 7.3. *The inclusion $C_c(\mathcal{G}_Y^Y) \subset C_c(\mathcal{G})$ extends to an isometric embedding from $C^*(\mathcal{G}_Y^Y)$ to $C^*(\mathcal{G})$. Also the inclusion $C_c(\mathcal{G}_Y^Y) \subset C_c(\mathcal{G})$ extends to an isometric embedding from $C_{red}^*(\mathcal{G}_Y^Y)$ to $C_{red}^*(\mathcal{G})$.*

Proof. Let $f \in C_c(\mathcal{G}_Y^Y)$ be given. Consider f as an element of $C_c(\mathcal{G}^Y) \subset \mathcal{E}$. Then $\theta_{f,f}$ restricted to $C_c(\mathcal{G}^Y)$ is just multiplication by $f * f^*$. Since \mathcal{E} is a $C^*(\mathcal{G})$ - $C^*(\mathcal{G}_Y^Y)$ imprimitivity bimodule, it follows that

$$\begin{aligned} \|f\|_{C^*(\mathcal{G})}^2 &= \|f * f^*\|_{C^*(\mathcal{G})} \\ &= \|\theta_{f,f}\| \\ &= \|f\|_{\mathcal{E}}^2 \\ &= \|f^* * f\|_{C^*(\mathcal{G}_Y^Y)} \\ &= \|f\|_{C^*(\mathcal{G}_Y^Y)}^2 \end{aligned}$$

For $x \in \mathcal{G}^0$, let $\mathcal{G}^{(x)} := r^{-1}(x)$. Consider $\ell^2(\mathcal{G}^{(x)})$ and let $\{\delta_\gamma : \gamma \in \mathcal{G}^{(x)}\}$ be the standard orthonormal basis. Consider the representation π_x of $C_c(\mathcal{G})$ on $\ell^2(\mathcal{G}^{(x)})$ defined by

$$(7.3) \quad \pi_x(f)(\delta_\gamma) = \sum_{\alpha \in \mathcal{G}^{(x)}} f(\alpha^{-1}\gamma) \delta_\alpha.$$

The reduced C^* -algebra $C_{red}^*(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ under the norm $\|\cdot\|$ given by $\|f\|_{red} = \sup_{x \in \mathcal{G}^0} \|\pi_x(f)\|$. (We refer the reader to [Ren09].)

Let $\mathcal{G}_Y^{(x)} := \{\gamma \in \mathcal{G}^{(x)} : s(\gamma) \in Y\}$. If $x \in Y$, let π_x^Y be the representation of $C_c(\mathcal{G}_Y^Y)$ on $\ell^2(\mathcal{G}_Y^{(x)})$ defined by the same formula as in Eq. 7.3. Now observe the following.

- (1) Let $\gamma_0 \in \mathcal{G}$ be such that $s(\gamma_0) = x$ and $r(\gamma_0) = y$. Then $U : \ell^2(\mathcal{G}^{(x)}) \rightarrow \ell^2(\mathcal{G}^{(y)})$ defined by $U(\delta_\gamma) = \delta_{\gamma_0\gamma}$ is a unitary. Moreover $U\pi_x(\cdot)U^* = \pi_y(\cdot)$.
- (2) Since Y meets each orbit of \mathcal{G}^0 , it follows from (1) that for $f \in C_c(\mathcal{G})$, $\|f\|_{red} = \sup_{x \in Y} \|\pi_x(f)\|$.
- (3) If $x \in Y$, then write $\ell^2(\mathcal{G}^{(x)})$ as $\ell^2(\mathcal{G}^{(x)}) = \ell^2(\mathcal{G}_Y^{(x)}) \oplus (\ell^2(\mathcal{G}_Y^{(x)}))^\perp$. With this decomposition, for $f \in C_c(\mathcal{G}_Y^Y)$, we have $\pi_x(f) = \pi_x^Y(f) \oplus 0$.

Now the above three observations imply that for $f \in C_c(\mathcal{G}_Y^Y)$, $\|f\|_{C_{red}^*(\mathcal{G}_Y^Y)} = \|f\|_{C_{red}^*(\mathcal{G})}$. This completes the proof. \square

Remark 7.4. *The representations used to define the regular representation in [Ren09] is different from what we have used. But the inversion map of the groupoid intertwines our representations with those used in [Ren09].*

The C^* -algebra of the groupoid $\overline{N} \rtimes G$ is naturally isomorphic to $C_0(\overline{N}) \rtimes G$. Let $\Phi : C_c(\overline{N}) \rtimes G \rightarrow C_c(\overline{N} \rtimes G)$ be the map defined by

$$(7.4) \quad \Phi(fU_g)(x, h) := \begin{cases} f(x) & \text{if } g = h, \\ 0 & \text{otherwise.} \end{cases}$$

for $f \in C_c(\overline{N})$ and $g \in G$. Here $\{U_g : g \in G\}$ denotes the canonical unitaries (corresponding to the group elements) in the multiplier algebra of $C_0(\overline{N}) \rtimes G$. Then Φ extends to an isomorphism from $C_0(\overline{N}) \rtimes G$ onto $C^*(\overline{N} \rtimes G)$ (Cf. Corollary 2.3.19, Page 34, [Ren09]).

Let $p := 1_{\overline{M}} \in C_c(\overline{N}) \subset C_0(\overline{N}) \rtimes G$ where $1_{\overline{M}}$ is the characteristic function associated to the compact open subset \overline{M} . Note that $\Phi(1_{\overline{M}}) = 1_{\overline{M} \times \{e\}}$.

Proposition 7.5. *The full corner $p(C_0(\overline{N}) \rtimes G)p$ is isomorphic to $\mathfrak{A}[N \rtimes H, M]$. Here the projection p is given by $p = 1_{\overline{M}}$.*

Proof. Let $i : C_c(\overline{N} \rtimes G|_{\overline{M}}) \rightarrow C_c(\overline{N} \rtimes G)$ be the natural inclusion. It is easy to verify that the image of i is $1_{\overline{M} \times \{e\}} C_c(\overline{N} \rtimes G) 1_{\overline{M} \times \{e\}}$. Now from Proposition 7.3, it follows that $C^*(\overline{N} \rtimes G|_{\overline{M}})$ is isomorphic to $1_{\overline{M} \times \{e\}} C^*(\overline{N} \rtimes G) 1_{\overline{M} \times \{e\}}$. But we have the isomorphism $\Phi : C_0(\overline{N}) \rtimes G \rightarrow C^*(\overline{N} \rtimes G)$ with $\Phi(1_{\overline{M}}) = 1_{\overline{M} \times \{e\}}$. Hence $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to the corner $1_{\overline{M}}(C_0(\overline{N}) \rtimes G) 1_{\overline{M}}$.

Let $A = C_0(\overline{N}) \rtimes G$. Then ApA is an ideal in A containing $p = 1_{\overline{M}}$. Note that for every $g \in G$, $x_g := U_g 1_{\overline{M}} 1_{\overline{M}} \in ApA$. Hence $1_{g\overline{M}} = U_g 1_{\overline{M}} U_g^* = x_g x_g^* \in ApA$. Hence for every $g \in G$, $1_{g\overline{M}} \in ApA$. Thus $1_{a^{-1}\overline{M}a} \in ApA$ for every $a \in P$. Thus we have $C_c(\overline{N}) \subset ApA$ (See Remark 7.6) and hence $C_0(\overline{N}) \subset ApA$. As a consequence we have $ApA = C_0(\overline{N}) \rtimes G$. Thus the projection p is full. This completes the proof. \square

Remark 7.6. *If $K \subset \overline{N}$ is compact then there exists $b \in P$ such that $K \subset b^{-1}\overline{M}b$. For $\{a^{-1}\overline{M}a : a \in P\}$ is an open cover of \overline{N} . Thus there exists $a_1, a_2, \dots, a_n \in P$ such that $K \subset \bigcup_{i=1}^n a_i^{-1}\overline{M}a_i$. Choose $b \in \bigcap_{i=1}^n Pa_i$. Then for every i , $a_i^{-1}\overline{M}a_i \subset b^{-1}\overline{M}b$. (Reason: M is dense in \overline{M} and $ba_i^{-1} \in P$). Hence $K \subset b^{-1}\overline{M}b$.*

Remark 7.7. *Using the second half of Proposition 7.3, it can be shown that the C^* -algebra $\mathfrak{A}_{red}[N \rtimes H, M]$ is isomorphic to the full corner $1_{\overline{M}}(C_0(\overline{N}) \rtimes_{red} G)1_{\overline{M}}$. We leave the details to the reader.*

8. CUNTZ-LI DUALITY THEOREM

The purpose of this section is to establish a duality result for the C^* -algebra associated to Examples 2.7 and 2.10. This is analogous to the duality result obtained in [CL11] for the ring C^* -algebra associated to the ring of integers in a number field. The proof is really a step by step adaptation of the arguments in [CL11] to our situation.

Let $\Gamma \subset GL_n(\mathbb{Q})$ be a subgroup and let $\Gamma_+ := \{\gamma \in \Gamma : \gamma \in M_n(\mathbb{Z})\}$. Assume that the following holds.

- (1) The group $\Gamma = \Gamma_+ \Gamma_+^{-1} = \Gamma_+^{-1} \Gamma_+$.
- (2) The intersections $\bigcap_{\gamma \in \Gamma_+} \gamma \mathbb{Z}^n = \bigcap_{\gamma \in \Gamma_+} \gamma^t \mathbb{Z}^n = \{0\}$.

Let $\Gamma^{op} := \{\gamma^t : \gamma \in \Gamma\}$. Then Γ^{op} is a subgroup of $GL_n(\mathbb{Q})$. Also Γ satisfies (1) and (2) if and only if Γ^{op} satisfies (1) and (2). If Γ contains the non-zero scalars then (1) and (2) are satisfied.

For the rest of this section, we let Γ be a subgroup of $GL_n(\mathbb{Q})$ which satisfies (1) and (2). The group Γ acts on \mathbb{Q}^n by left multiplication. Let $N_\Gamma := \bigcup_{\gamma \in \Gamma_+} \gamma^{-1} \mathbb{Z}^n$. Then by Lemma 2.3, it follows that N_Γ is a subgroup of \mathbb{Q}^n and Γ leaves N_Γ invariant. Consider the semidirect product $N_\Gamma \rtimes \Gamma$. Then the pair $(N_\Gamma \rtimes \Gamma, \mathbb{Z}^n)$ satisfies the hypotheses (C1), (C2) and (C3). Let us denote the C^* -algebra $\mathfrak{A}[N_\Gamma \rtimes \Gamma, \mathbb{Z}^n]$ by \mathfrak{A}_Γ .

Note that $N_\Gamma \rtimes \Gamma$ acts on \mathbb{R}^n on the right as follows. For $\xi \in \mathbb{R}^n$ and $(v, \gamma) \in N_\Gamma \rtimes \Gamma$, let $\xi \cdot (v, \gamma) = \gamma^{-1}(\xi - v)$. This right action of $N_\Gamma \rtimes \Gamma$ on \mathbb{R}^n gives rise to a left action of $N_\Gamma \rtimes \Gamma$ on $C_0(\mathbb{R}^n)$ as follows. For $g \in N_\Gamma \rtimes \Gamma$ and $f \in C_0(\mathbb{R}^n)$, let $(g \cdot f)(x) = f(x \cdot g)$.

The main theorem of this section is the following.

Theorem 8.1. *The C^* -algebras $\mathfrak{A}_{\Gamma^{op}}$ and $C_0(\mathbb{R}^n) \rtimes (N_\Gamma \rtimes \Gamma)$ are Morita-equivalent.*

To prove this we need a bit of preparation. If $\gamma \in \Gamma_+$, then γ leaves \mathbb{Z}^n invariant and induces a map on the quotient $\frac{N_\Gamma}{\mathbb{Z}^n}$ which we still denote by γ . Let

$$\overline{N}_\Gamma := \{(z_\gamma)_{\gamma \in \Gamma_+} \in \prod_{\gamma \in \Gamma_+} \frac{N_\Gamma}{\mathbb{Z}^n} : \delta z_{\gamma\delta} = z_\gamma \text{ for every } \gamma, \delta \in \Gamma_+\}$$

We give $\frac{N_\Gamma}{\mathbb{Z}^n}$ the discrete topology. The abelian group $\overline{N_\Gamma}$ is given the subspace topology inherited from the product topology on $\prod_{\gamma \in \Gamma_+} \frac{N_\Gamma}{\mathbb{Z}^n}$. The topological group $\overline{N_\Gamma}$ is Hausdorff.

Now we describe the action of Γ_+ on $\overline{N_\Gamma}$. Let $\gamma \in \Gamma_+$ and $z \in \overline{N_\Gamma}$ be given. For $\delta \in \Gamma_+$, choose $\alpha, \beta \in \Gamma_+$ such that $\gamma\alpha = \delta\beta$. Let $(\gamma.z)_\delta = \beta z_\alpha$. It is easily verifiable that γ is a homeomorphism. The inverse of γ is given by $(\gamma^{-1}z)_\delta = z_{\gamma\delta}$. This way Γ_+ acts on $\overline{N_\Gamma}$ and induces an action of Γ on $\overline{N_\Gamma}$.

Proposition 8.2. *We have the following.*

- (1) *The map $N_\Gamma \ni v \rightarrow (\gamma^{-1}v)_{\gamma \in \Gamma_+} \in \overline{N_\Gamma}$ is injective and is Γ -equivariant. Moreover, when N_Γ is viewed as a subset of $\overline{N_\Gamma}$ via this embedding, N_Γ is dense in $\overline{N_\Gamma}$.*
- (2) *Let $\overline{M_\Gamma} := \{z \in \overline{N_\Gamma} : z_e = 0\}$ is a compact open subgroup of $\overline{N_\Gamma}$. Also the intersection $\overline{M_\Gamma} \cap N_\Gamma = \mathbb{Z}^n$. Hence \mathbb{Z}^n is dense in $\overline{M_\Gamma}$.*
- (3) *Also $\overline{N_\Gamma} = \bigcup_{\gamma \in \Gamma_+} \gamma^{-1}\overline{M_\Gamma}$. As a consequence, $\overline{N_\Gamma}$ is locally compact.*

Proof. The fact that $v \rightarrow (\gamma^{-1}v)_\gamma$ is injective follows from the assumption that $\bigcap_{\gamma \in \Gamma_+} \gamma\mathbb{Z}^n = \{0\}$. Let $\gamma \in \Gamma_+$ and $v \in N_\Gamma$ be given. Let us denote the image of v in $\overline{N_\Gamma}$ by \tilde{v} . We need to show that for $\delta \in \Gamma_+$, the δ^{th} co-ordinate of $\gamma.\tilde{v}$ is $\delta^{-1}\gamma v$. Choose α and β in Γ_+ such that $\gamma\alpha = \delta\beta$. Then by definition $(\gamma.\tilde{v})_\delta = \beta\alpha^{-1}v = \delta^{-1}\gamma v$. Thus we have shown that the embedding $N_\Gamma \ni v \rightarrow (\gamma^{-1}v)_{\gamma \in \Gamma_+} \in \overline{N_\Gamma}$ is Γ_+ -equivariant and consequently is Γ -equivariant.

For $\gamma \in \Gamma_+$ and $v \in N_\Gamma$, let

$$U_{\gamma,v} := \{z \in \overline{N_\Gamma} : z_\gamma \equiv v \pmod{\mathbb{Z}^n}\}.$$

Clearly the collection $\{U_{\gamma,v} : \gamma \in \Gamma_+, v \in N_\Gamma\}$ forms a basis for $\overline{N_\Gamma}$. Note that $\gamma.v \in U_{\gamma,v}$. Thus N_Γ is dense in $\overline{N_\Gamma}$.

For $\gamma \in \Gamma_+$, let $N_\gamma := \gamma^{-1}\mathbb{Z}^n$. Note that for $\gamma \in \Gamma_+$, $\frac{N_\gamma}{\mathbb{Z}^n}$ is finite. Now observe that $\overline{M_\Gamma} = \overline{N_\Gamma} \cap \prod_\gamma \frac{N_\gamma}{\mathbb{Z}^n}$. Thus $\overline{M_\Gamma}$ is compact. Since the projection onto the e^{th} co-ordinate is a continuous homomorphism, it follows that $\overline{M_\Gamma}$ is an open subgroup. The equality $\overline{M_\Gamma} \cap N_\Gamma = \mathbb{Z}^n$ is obvious.

Let $z \in \overline{N_\Gamma}$ be given. Since $N_\Gamma = \bigcup_{\gamma \in \Gamma_+} \gamma^{-1}\mathbb{Z}^n$, it follows that there exists $\gamma \in \Gamma_+$ such that $\gamma z_e = 0$. Then $\gamma.z \in \overline{M_\Gamma}$. Thus $\overline{N_\Gamma} = \bigcup_{\gamma \in \Gamma_+} \gamma^{-1}\overline{M_\Gamma}$. As $\overline{N_\Gamma}$ is a union of compact open subsets, it follows that $\overline{N_\Gamma}$ is locally compact. This completes the proof. \square

Let $\overline{N'}$ and $\overline{M'}$ be the groups considered in Section 6 applied to the pair $(N_\Gamma \rtimes \Gamma, \mathbb{Z}^n)$. Let us now convince ourselves that the pair $(\overline{N'}, \overline{M'})$ is Γ -equivariantly isomorphic to the pair $(\overline{N_\Gamma}, \overline{M_\Gamma})$. Let $\gamma, \delta \in \Gamma_+$ be given.

Denote the quotient map $\mathbb{Z}^n \rightarrow \frac{\mathbb{Z}^n}{\gamma\mathbb{Z}^n}$ by q_γ . Then q_γ descends to a map $\frac{\mathbb{Z}^n}{\gamma\delta\mathbb{Z}^n} \rightarrow \frac{\mathbb{Z}^n}{\gamma\mathbb{Z}^n}$ which we denote by $q_{\gamma,\delta}$. Multiplication by γ^{-1} maps \mathbb{Z}^n injectively onto $\gamma^{-1}\mathbb{Z}^n$ and takes $\gamma\mathbb{Z}^n$ onto \mathbb{Z}^n . We denote the resulting isomorphism from $\frac{\mathbb{Z}^n}{\gamma\mathbb{Z}^n} \rightarrow \frac{\gamma^{-1}\mathbb{Z}^n}{\mathbb{Z}^n}$ again by γ^{-1} . Then we have the following commutative diagram where the vertical arrows are isomorphisms.

$$(8.5) \quad \begin{array}{ccc} \frac{\mathbb{Z}^n}{\gamma\delta\mathbb{Z}^n} & \xrightarrow{q_{\gamma,\delta}} & \frac{\mathbb{Z}^n}{\gamma\mathbb{Z}^n} \\ \downarrow (\gamma\delta)^{-1} & & \downarrow \gamma^{-1} \\ \frac{(\gamma\delta)^{-1}\mathbb{Z}^n}{\mathbb{Z}^n} & \xrightarrow{\delta} & \frac{\gamma^{-1}\mathbb{Z}^n}{\mathbb{Z}^n} \end{array}$$

Recall that

$$\begin{aligned} \overline{M'} &= \{(z_\gamma)_{\gamma \in \Gamma_+} \in \prod_{\gamma \in \Gamma_+} \frac{\mathbb{Z}^n}{\gamma\mathbb{Z}^n} : q_{\gamma,\delta}(z_{\gamma\delta}) = z_\gamma\} \\ \overline{M_\Gamma} &= \{(z_\gamma)_{\gamma \in \Gamma_+} \in \prod_{\gamma \in \Gamma_+} \frac{\gamma^{-1}\mathbb{Z}^n}{\mathbb{Z}^n} : \delta z_{\gamma\delta} = z_\gamma\} \end{aligned}$$

Let $i : \mathbb{Z}^n \rightarrow \overline{M'}$ be the embedding given by $i(v) = (v)_{\gamma \in \Gamma_+}$ and $j : \mathbb{Z}^n \rightarrow \overline{M_\Gamma}$ be the embedding described in Proposition 8.2. Then $j(v) = (\gamma^{-1}v)_{\gamma \in \Gamma_+}$ for $v \in \mathbb{Z}^n$. Now the commutative diagram 8.5 implies that the map $\varphi : \overline{M'} \rightarrow \overline{M_\Gamma}$ given by $\varphi((z_\gamma)) = (\gamma^{-1}z_\gamma)$ is an isomorphism and $\varphi(i(v)) = j(v)$ for $v \in \mathbb{Z}^n$. It is also clear that φ is a homeomorphism.

Claim: φ is Γ_+ -equivariant. First the embeddings i and j are Γ_+ -equivariant. Since $\varphi \circ i = j$, it follows that $\varphi(\gamma.i(v)) = \gamma.\varphi(i(v))$ if $\gamma \in \Gamma_+$ and $v \in \mathbb{Z}^n$. Since $i(\mathbb{Z}^n)$ is dense in $\overline{M'}$ (and the maps involved are continuous), it follows that $\varphi(\gamma.x) = \gamma.\varphi(x)$ for $x \in \overline{M'}$ and $\gamma \in \Gamma_+$.

Now since $\overline{N_\Gamma} = \bigcup_{\gamma \in \Gamma_+} \gamma^{-1}\overline{M_\Gamma}$ and $\overline{N'} = \bigcup_{\gamma \in \Gamma_+} \gamma^{-1}\overline{M'}$, it follows from the universal property, as explained in Section 6 (item 9), that the map $\gamma^{-1}x \rightarrow \gamma^{-1}\varphi(x)$ (with

$x \in \overline{M'})$ extends to a Γ -equivariant isomorphism from $\overline{N'} \rightarrow \overline{N_\Gamma}$. \square

Now we describe the Pontryagin dual of the discrete group N_Γ . For $x, \xi \in \mathbb{R}^n$, let $\langle x, \xi \rangle := x^t \xi$. If $x, \xi \in \mathbb{R}^n$, we let $\chi_\xi(x) = e^{2\pi i \langle x, \xi \rangle}$. We identify \mathbb{R}^n with $\widehat{\mathbb{R}^n}$ via the map $\xi \rightarrow \chi_\xi$. If $\xi \in \mathbb{R}^n$, restricting χ_ξ to N_Γ gives a character of N_Γ . Moreover the map $\mathbb{R}^n \ni \xi \rightarrow \chi_\xi \in \widehat{N_\Gamma}$ is continuous.

Let $z \in \overline{N_{\Gamma^{op}}}$ be given. Let $\chi_z : N_\Gamma \rightarrow \mathbb{T}$ be defined as follows. For $x \in \gamma^{-1}\mathbb{Z}^n$ for some $\gamma \in \Gamma_+$, let $\chi_z(x) = e^{2\pi i \langle \gamma x, z_\gamma \rangle} = e^{2\pi i \langle x, \gamma^t z_\gamma \rangle}$. It is easy to verify that χ_z is well defined and χ_z is a character of N_Γ . Clearly $\overline{N_{\Gamma^{op}}} \ni z \rightarrow \chi_z \in \widehat{N_\Gamma}$ is continuous. Note that if $z \in N_{\Gamma^{op}}$ and $x \in N_\Gamma$ then $\chi_z(x) = e^{2\pi i \langle x, z \rangle}$.

Proposition 8.3. *The map $\Psi : \mathbb{R}^n \times \overline{N_{\Gamma^{op}}} \rightarrow \widehat{N_\Gamma}$ defined by*

$$\Psi(\xi, z) = \chi_\xi \chi_{-z}$$

is a surjective homomorphism with kernel $\Delta = \{(x, x) : x \in N_{\Gamma^{op}}\}$. The induced map $\tilde{\Psi} : \frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta} \rightarrow \widehat{N_\Gamma}$ is a topological isomorphism.

Proof. Clearly Ψ is a continuous group homomorphism and $\Psi(\Delta) = \{1\}$. Now let us show that the kernel of Ψ is Δ . Let (ξ, z) be such that $\Psi(\xi, z) = 1$. Then for every $\gamma \in \Gamma_+$ and $x \in \mathbb{Z}^n$, we have

$$\begin{aligned} 1 &= \chi_\xi(\gamma^{-1}x) \chi_{-z}(\gamma^{-1}x) \\ &= e^{2\pi i \langle x, (\gamma^t)^{-1}\xi \rangle} e^{-2\pi i \langle x, z_\gamma \rangle} \\ &= e^{2\pi i \langle x, (\gamma^t)^{-1}\xi - z_\gamma \rangle} \end{aligned}$$

Thus for every $\gamma \in \Gamma_+$, we have $z_\gamma - (\gamma^t)^{-1}\xi \in \mathbb{Z}^n$. In other words, we have $\xi \in N_{\Gamma^{op}}$ and $z = \xi$ in $\overline{N_{\Gamma^{op}}}$. Hence $(\xi, z) \in \Delta$. Thus we have shown that the kernel of Ψ is Δ which implies that $\tilde{\Psi}$ is one-one.

Next we claim $\frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$ is compact. Let $\lambda : \mathbb{R}^n \times \overline{N_{\Gamma^{op}}} \rightarrow \frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$ be the quotient map. We also write $\lambda(\xi, z)$ as $[(\xi, z)]$. We claim that $\lambda([0, 1]^n \times \overline{M_{\Gamma^{op}}}) = \frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$. This will prove that $\frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$ is compact.

Let $[(\xi, z)]$ be an element in the quotient $\frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$. Choose $v \in \mathbb{Z}^n$ and $\gamma \in \Gamma_+$ such that $z_e \equiv (\gamma^t)^{-1}v$. Then $[(\xi, z)] = [(\xi - (\gamma^t)^{-1}v, z - (\gamma^t)^{-1}v)]$. Choose $w \in \mathbb{Z}^n$ such that $\xi - (\gamma^t)^{-1}v - w \in [0, 1]^n$. Let $\xi' = \xi - (\gamma^t)^{-1}v - w$ and $z' = z - (\gamma^t)^{-1}v - w$. Then $\xi' \in [0, 1]^n$ and $z' \in \overline{M_{\Gamma^{op}}}$. Moreover $\lambda(\xi, z) = \lambda(\xi', z')$. Thus the image of $[0, 1]^n \times \overline{M_{\Gamma^{op}}}$ under λ is $\frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$.

The image of $\widetilde{\Psi}$ is a compact subgroup of $\widehat{N_\Gamma}$ and it separates points of N_Γ (The image of $\mathbb{R}^n \times \{0\}$ under Ψ separates points of N_Γ). Hence $\widetilde{\Psi}$ is onto. Since $\frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$ is compact, it follows that $\widetilde{\Psi}$ is a topological isomorphism. This completes the proof. \square

Consider the semidirect product $\mathbb{R}^n \rtimes \Gamma^{op}$ where Γ^{op} acts on \mathbb{R}^n by left multiplication. The semidirect product $\mathbb{R}^n \rtimes \Gamma^{op}$ acts on $\widehat{N_\Gamma} = \frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$ on the right as follows. For $[(\xi, z)] \in \widehat{N_\Gamma}$ and $(v, \gamma) \in \mathbb{R}^n \rtimes \Gamma^{op}$, let $[(\xi, z)] \cdot (v, \gamma) = [(\gamma^{-1}(\xi + v), \gamma^{-1}z)]$. This right action of $\mathbb{R}^n \rtimes \Gamma^{op}$ on $\widehat{N_\Gamma}$ induces a left action of $\mathbb{R}^n \rtimes \Gamma^{op}$ on $C^*(N_\Gamma) \cong C(\widehat{N_\Gamma})$.

The crossed product $C^*(N_\Gamma) \rtimes (\mathbb{R}^n \rtimes \Gamma^{op})$ is isomorphic to the iterated crossed product $(C^*(N_\Gamma) \rtimes \mathbb{R}^n) \rtimes \Gamma^{op}$. (Cf. Proposition 3.11, Page 87, [Wil07].) But then the map $\Gamma \ni \gamma \rightarrow (\gamma^t)^{-1} \in \Gamma^{op}$ is an isomorphism. Thus the crossed product $(C^*(N_\Gamma) \rtimes \mathbb{R}^n) \rtimes \Gamma^{op} \cong (C^*(N_\Gamma) \rtimes \mathbb{R}^n) \rtimes \Gamma$.

Let us fix notations. Let τ be the action of \mathbb{R}^n on $C^*(N_\Gamma)$. Let β be the action of Γ on $C^*(N_\Gamma) \cong C(\widehat{N_\Gamma})$, induced by the action of Γ^{op} and the identification $\Gamma \cong \Gamma^{op}$. For $v \in N_\Gamma$, $\xi \in \mathbb{R}^n$ and $\gamma \in \Gamma$, it is easy to verify the following.

$$\tau_\xi(\delta_v) = e^{-2\pi i \langle \xi, v \rangle} \delta_v,$$

$$\beta_\gamma(\delta_v) = \delta_{\gamma v}$$

where $\{\delta_v : v \in N_\Gamma\}$ denotes the canonical unitaries of $C^*(N_\Gamma)$. The action of Γ^{op} on $C^*(N_\Gamma) \rtimes \mathbb{R}^n$, induces an action of Γ (via the identification $\Gamma \ni \gamma \rightarrow (\gamma^t)^{-1}$) and let us denote it by $\widetilde{\beta}$. For $\gamma \in \Gamma$, and $f \in C_c(\mathbb{R}^n, C^*(N_\Gamma))$, we have

$$\widetilde{\beta}_\gamma(f)(x) = |\det(\gamma)| \beta_\gamma(f(\gamma^t x)).$$

Now consider the crossed product $C_0(\mathbb{R}^n) \rtimes (N_\Gamma \rtimes \Gamma) \cong C^*(\mathbb{R}^n) \rtimes (N_\Gamma \rtimes \Gamma)$. Let us denote the action of N_Γ and Γ on $C^*(\mathbb{R}^n)$ by σ and α . For $v \in N_\Gamma$, $\gamma \in \Gamma$ and $f \in C_c(\mathbb{R}^n)$, we have

$$(\sigma_v f)(\xi) = e^{2\pi i \langle \xi, v \rangle} f(\xi),$$

$$(\alpha_\gamma f)(\xi) = |\det(\gamma)| f(\gamma^t \xi).$$

Denote the action of Γ on $C^*(\mathbb{R}^n) \rtimes N_\Gamma$ by $\widetilde{\alpha}$. For $\gamma \in \Gamma$, $v \in N_\Gamma$ and $f \in C^*(\mathbb{R}^n)$, one has

$$\widetilde{\alpha}_\gamma(f \delta_v) = \alpha_\gamma(f) \delta_{\gamma v}.$$

Let us recall the following lemma which is Lemma 4.3 in [CL11].

Lemma 8.4 ([CL11]). *Let G be a locally compact abelian group and H be a subgroup of the Pontryagin dual \widehat{G} . Endow H with the discrete topology. Let σ be the action of*

H on $C^*(G)$ and τ be the action of G on $C^*(H)$ given by $\sigma_h(f) = [g \rightarrow h(g)f(g)]$ and $\tau_g(\tilde{f}) = [h \rightarrow h(-g)\tilde{f}(h)]$. Then the map $\phi : C_c(H, C_c(G)) \rightarrow C_c(G, C_c(H))$ defined by $\phi(f)(g)(h) = h(-g)f(h)(g)$ extends to an isomorphism between $C^*(G) \rtimes_\sigma H$ and $C^*(H) \rtimes_\tau G$.

We are now ready to prove the following proposition.

Proposition 8.5. *The crossed products $C_0(\mathbb{R}^n) \rtimes (N_\Gamma \rtimes \Gamma)$ and $C(\widehat{N_\Gamma}) \rtimes (\mathbb{R}^n \rtimes \Gamma^{op})$ are isomorphic.*

Proof. It is enough to show that the crossed products $(C^*(\mathbb{R}^n) \rtimes_\sigma N_\Gamma) \rtimes_{\tilde{\alpha}} \Gamma$ and $(C^*(N_\Gamma) \rtimes_\tau \mathbb{R}^n) \rtimes_{\tilde{\beta}} \Gamma$ are isomorphic. We show that $C^*(\mathbb{R}^n) \rtimes_\sigma N_\Gamma$ and $C^*(N_\Gamma) \rtimes_\tau \mathbb{R}^n$ are Γ -equivariantly isomorphic. Then the isomorphism between the crossed products will follow.

Identify \mathbb{R}^n with $\widehat{\mathbb{R}^n}$ via the map $\xi \rightarrow \chi_\xi$. (Recall that χ_ξ is the character given by $\chi_\xi(x) = e^{2\pi i \langle x, \xi \rangle}$.) Consider N_Γ as a subgroup of $\widehat{\mathbb{R}^n}$ via the natural inclusion $N_\Gamma \subset \mathbb{R}^n$. Note that the action σ of N_Γ on $C^*(\mathbb{R}^n)$ and τ of \mathbb{R}^n on $C^*(N_\Gamma)$ are exactly as in Lemma 8.4.

Thus Lemma 8.4 implies that $C^*(\mathbb{R}^n) \rtimes_\sigma N_\Gamma \cong C^*(N_\Gamma) \rtimes_\tau \mathbb{R}^n$. Let $\phi : C^*(\mathbb{R}^n) \rtimes_\sigma N_\Gamma \rightarrow C^*(N_\Gamma) \rtimes_\tau \mathbb{R}^n$ be the isomorphism prescribed by Lemma 8.4. We claim ϕ is Γ -equivariant. First note that $\phi(f\delta_v)(\xi) = e^{-2\pi i \langle \xi, v \rangle} f(\xi)\delta_v$ for $f \in C_c(\mathbb{R}^n)$ and $v \in N_\Gamma$.

Let $\gamma \in \Gamma$ be given. Now observe that

$$\begin{aligned} \tilde{\beta}_\gamma(\phi(f\delta_v))(\xi) &= |\det(\gamma)|\beta_\gamma(\phi(f\delta_v)(\gamma^t\xi)) \\ &= |\det(\gamma)|e^{-2\pi i \langle \gamma^t\xi, v \rangle} f(\gamma^t\xi)\delta_{\gamma v} \\ &= |\det(\gamma)|e^{-2\pi i \langle \xi, \gamma v \rangle} f(\gamma^t\xi)\delta_{\gamma v}. \end{aligned}$$

On the other hand, observe that

$$\begin{aligned} \phi(\widetilde{\alpha_\gamma}(f\delta_v))(\xi) &= \phi(\alpha_\gamma(f)\delta_{\gamma v})(\xi) \\ &= e^{-2\pi i \langle \xi, \gamma v \rangle} \alpha_\gamma(f)(\xi)\delta_{\gamma v} \\ &= e^{-2\pi i \langle \xi, \gamma v \rangle} |\det(\gamma)|f(\gamma^t\xi)\delta_{\gamma v} \end{aligned}$$

Hence for every $\gamma \in \Gamma$, $\tilde{\beta}_\gamma\phi(f\delta_v) = \phi\widetilde{\alpha_\gamma}(f\delta_v)$. Since $\{f\delta_v : f \in C_c(\mathbb{R}^n), v \in N_\Gamma\}$ is total in $C^*(\mathbb{R}^n) \rtimes_\sigma N_\Gamma$, it follows that for every γ , $\tilde{\beta}_\gamma\phi = \phi\widetilde{\alpha_\gamma}$. In other words, ϕ is Γ -equivariant. This completes the proof. \square

Proof of Theorem 8.1. By Corollary 6.6, it follows that $\mathfrak{A}_{\Gamma^{op}}$ is isomorphic to the C^* -algebra of the groupoid $\tilde{\mathcal{G}} := \overline{N}_{\Gamma^{op}} \rtimes (N_{\Gamma^{op}} \rtimes \Gamma^{op})|_{\overline{M}_{\Gamma^{op}}}$. By Proposition 8.5, it follows that

$C_0(\mathbb{R}^n) \rtimes (N_\Gamma \rtimes \Gamma)$ is isomorphic to the C^* -algebra of the groupoid $\mathcal{G} := \widehat{N_\Gamma} \rtimes (\mathbb{R}^n \rtimes \Gamma^{op})$. We will show that \mathcal{G} and $\tilde{\mathcal{G}}$ are equivalent in the sense of [MRW87].

By Proposition 8.3, $\widehat{N_\Gamma} = \frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$ where $\Delta := \{(x, x) : x \in N_{\Gamma^{op}}\}$. Denote the quotient map $\mathbb{R}^n \times \overline{N_{\Gamma^{op}}} \rightarrow \frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$ by λ . Let $X := \lambda(\{0\} \times \overline{M_{\Gamma^{op}}})$. Then X is a closed subset of \mathcal{G}^0 and it is easy to verify that X meets each orbit of \mathcal{G}^0 . Let

$$\mathcal{G}_X := \{\alpha \in \mathcal{G} : s(\alpha) \in X\} = s^{-1}(X)$$

We claim that the (restricted) source map $s : \mathcal{G}_X \rightarrow X$ and the range map $r : \mathcal{G}_X \rightarrow \mathcal{G}^0$ are open. Let $U \subset \mathcal{G}$ be an open subset. Then $s(U \cap \mathcal{G}_X) = s(U) \cap X$. Since $s : \mathcal{G} \rightarrow \mathcal{G}^0$ is open, it follows that $s : \mathcal{G}_X \rightarrow X$ is open.

Now we prove that $r : \mathcal{G}_X \rightarrow \mathcal{G}^0$ is open. It is enough to show that $r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X)$ is open whenever $U \subset \frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$ and $V \subset \mathbb{R}^n$ are open and $\gamma \in \Gamma^{op}$. We claim that

$$r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X) = U \cap \lambda(-V \times \gamma \overline{M_{\Gamma^{op}}})$$

Let $[(\xi, z)] \in r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X)$. Then there exists $[(\eta, y)], (v, \gamma) \in U \times V \times \{\gamma\}$ such that $[(\eta, y)].(v, \gamma) \in X$ and $[(\xi, z)] = [(\eta, y)]$. Thus there exists $u \in N_{\Gamma^{op}}$ such that $\gamma^{-1}(\xi + v) = u$ and $\gamma^{-1}z - u = x$ for some $x \in \overline{M_{\Gamma^{op}}}$. Hence $[(\xi, z)] = [(-v, \gamma x)]$. Clearly $[(\xi, z)] \in U$. Hence $[(\xi, z)] \in U \cap \lambda(-V \times \gamma \overline{M_{\Gamma^{op}}})$. Thus we have shown that

$$r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X) \subset U \cap \lambda(-V \times \gamma \overline{M_{\Gamma^{op}}}).$$

Now let $[(\xi, z)] \in U \cap \lambda(-V \times \gamma \overline{M_{\Gamma^{op}}})$. Then there exists $(v, x) \in V \times \overline{M_{\Gamma^{op}}}$ such that $[(\xi, z)] = [(-v, \gamma x)]$. This is equivalent to saying that $[(\xi, z)].(v, \gamma) \in X$. Thus $[(\xi, z)], (v, \gamma) \in (U \times V \times \{\gamma\}) \cap \mathcal{G}_X$ and $r([(\xi, z)], (v, \gamma)) = (\xi, z)$. This proves that $U \cap \lambda(-V \times \gamma \overline{M_{\Gamma^{op}}}) \subset r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X)$.

This proves the claim that $r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X) = U \cap \lambda(-V \times \gamma \overline{M_{\Gamma^{op}}})$. Now since λ is open and $\overline{M_{\Gamma^{op}}}$ is open, it follows that $r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X)$ is open. Thus we have shown that $r : \mathcal{G}_X \rightarrow \mathcal{G}^0$ is open.

Now by Example 2.7 of [MRW87], it follows that \mathcal{G} and $\mathcal{G}_X^X := \{\alpha \in \mathcal{G}_X : r(\alpha) \in X\}$ are equivalent. Recall that $\tilde{\mathcal{G}} = \overline{N_{\Gamma^{op}}} \rtimes (N_{\Gamma^{op}} \rtimes \Gamma^{op})|_{\overline{M_{\Gamma^{op}}}}$. The right action of $N_{\Gamma^{op}} \rtimes \Gamma^{op}$ on $\overline{N_{\Gamma^{op}}}$ is given by $x.(v, \gamma) = \gamma^{-1}(x - v)$. Let $\Phi : \tilde{\mathcal{G}} \rightarrow \mathcal{G}_X^X$ be defined by $\Phi(x, v, \gamma) = ([(0, x)], v, \gamma)$. It is easy to check that Φ is a groupoid isomorphism and it is continuous. Now we prove that Φ is a topological isomorphism.

Let (x_n, v_n, γ) be a sequence in $\tilde{\mathcal{G}}$ such that $\Phi(x_n, v_n, \gamma)$ converges to $([(0, x)], v, \gamma)$. First note that $x \rightarrow [(0, x)]$ is a topological embedding of $\overline{M_{\Gamma^{op}}}$ into $\widehat{N_\Gamma}$. Thus, it follows that x_n converges to x in $\overline{M_{\Gamma^{op}}}$. Now $\Phi(x_n, v_n, \gamma)$ converges to $[(0, x)], v, \gamma$ implies that

v_n tends to v in \mathbb{R}^n and $\gamma^{-1}(x - v_n)$ tends to $\gamma^{-1}(x - v)$ in $\overline{M}_{\Gamma^{op}}$. Hence v_n converges to v in $\overline{N}_{\Gamma^{op}}$. Thus $(v_n, v_n) \rightarrow (v, v)$ in $\mathbb{R}^n \times \overline{N}_{\Gamma^{op}}$. But Δ is a discrete subgroup of $\mathbb{R}^n \times \overline{N}_{\Gamma^{op}}$. Hence $v_n = v$ eventually. Therefore, $(x_n, v_n, \gamma) \rightarrow (x, v, \gamma)$ in $\tilde{\mathcal{G}}$. So, Φ is a topological isomorphism.

Since \mathcal{G} and $\tilde{\mathcal{G}}$ are equivalent in the sense of [MRW87], it follows from Theorem 2.8 in [MRW87] that $C^*(\mathcal{G})$ and $C^*(\tilde{\mathcal{G}})$ are Morita-equivalent. This completes the proof. \square

8.1. Examples. We end this article by considering two examples.

Example 1: First we show that the duality result for the ring C^* -algebra associated to number fields obtained in [CL11] can be derived from Theorem 8.1.

Consider a number field K of degree n . Denote the ring of integers in K by O_K . Let $\{w_1, w_2, \dots, w_n\}$ be a \mathbb{Z} -basis for O_K . Then $\{w_1, w_2, \dots, w_n\}$ is a \mathbb{Q} -basis for K . Identify K with \mathbb{Q}^n via the map $\beta : \mathbb{Q}^n \ni (x_1, x_2, \dots, x_n)^t \rightarrow \sum_{i=1}^n x_i w_i \in K$. By definition, $\beta(\mathbb{Z}^n) = O_K$.

If $a \in K$, then a acts on K by left multiplication and is \mathbb{Q} -linear. Thus a gives rise to a matrix with respect to the basis $\{w_1, w_2, \dots, w_n\}$ which we denote by $\alpha(a)$. Explicitly, for $1 \leq j \leq n$, let

$$(8.6) \quad aw_j := \sum_{i=1}^n \alpha_{ij}(a)w_i.$$

Let $\alpha(a) := (\alpha_{ij}(a))$. Then $\alpha : K \rightarrow M_n(\mathbb{Q})$ is an injective ring homomorphism. We also have the following equivariance. For $a \in K$ and $x \in \mathbb{Q}^n$, $\beta(\alpha(a)x) = a\beta(x)$.

Let $\Gamma := \alpha(K^\times)$. Then Γ is a subgroup of $GL_n(\mathbb{Q})$. Now the pair $(K \rtimes K^\times, O_K)$ is isomorphic to $(\mathbb{Q}^n \rtimes \Gamma, \mathbb{Z}^n)$. Thus the ring C^* -algebra associated to O_K is nothing but $\mathfrak{A}[\mathbb{Q}^n \rtimes \Gamma, \mathbb{Z}^n]$. Hence Theorem 8.1 applies. The only thing that one needs to verify is $\bigcap_{a \in O_K} \alpha(a)^t \mathbb{Z}^n$ is trivial. Since $\bigcap_{a \in O_K} aO_K = \{0\}$, it follows that $\bigcap_{a \in O_K} \alpha(a)\mathbb{Z}^n = \{0\}$. We produce a matrix X with rational entries whose determinant is non-zero and $X\alpha(a)X^{-1} = \alpha(a)^t$ for every $a \in O_K$. Then it will follow that $\bigcap_{a \in O_K} \alpha(a)^t \mathbb{Z}^n = \{0\}$. (See also Lemma 8.8.)

Let $Tr : M_n(\mathbb{Q}) \rightarrow \mathbb{Q}$ be the usual trace and let $tr := Tr \circ \alpha$. Denote the $n \times n$ matrix whose $(i, j)^{\text{th}}$ entry is $tr(w_i w_j)$ by X . Then X has determinant non-zero and its determinant is called the discriminant of the number field K .

Lemma 8.6. *For every $a \in K$, $X\alpha(a)X^{-1} = \alpha(a)^t$.*

Proof. Fix $a \in K$. Let $Y = (tr(aw_iw_j))$. Multiplying Equation 8.6 by w_k and taking trace, we get

$$Y_{jk} = \sum_{i=1}^n \alpha_{ij}(a) X_{ik}$$

In other words, we have $Y = \alpha(a)^t X$. But Y and X are symmetric. Thus taking transpose, we get $Y = X\alpha(a)$. Hence $X\alpha(a) = \alpha(a)^t X$. This completes the proof. \square

Let \mathbb{A}_∞ denote the ring of infinite adeles associated to K .

Theorem 8.7 ([CL11]). *For a number field K , the ring C^* -algebra $\mathfrak{A}[K \rtimes K^\times, O_K]$ is Morita-equivalent to $C_0(\mathbb{A}_\infty) \rtimes (K \rtimes K^\times)$.*

Proof. Note that for $\Gamma = \alpha(K^\times)$, $N_\Gamma = \mathbb{Q}^n$ and $N_{\Gamma^{op}} = \mathbb{Q}^n$ (since Γ contains the diagonal matrices with rational entries). Thus Lemma 8.6 implies that the matrix $X = (tr(w_iw_j))$ implements an isomorphism between the dynamical systems $(\mathbb{R}^n, \mathbb{Q}^n \rtimes \Gamma)$ and $(\mathbb{R}^n, \mathbb{Q}^n \rtimes \Gamma^{op})$. The map

$$(\mathbb{R}^n, \mathbb{Q}^n \rtimes \Gamma) \ni (\xi, (v, \gamma)) \rightarrow (X\xi, (Xv, \gamma^t)) \in (\mathbb{R}^n, \mathbb{Q}^n \rtimes \Gamma^{op})$$

is the required isomorphism. (Note that Γ is commutative.)

Consider the map $\delta : \mathbb{R}^n \ni (x_1, x_2, \dots, x_n) \rightarrow \sum_{i=1}^n x_i w_i \in \mathbb{A}_\infty$. Then from standard number theoretic arguments, (for example, using Theorem 13.5 (page 70) and Theorem 4.4 (page 110) in [Jan96]), it follows that δ (together with identifications α and β) implements an isomorphism between $(\mathbb{A}_\infty, K \times K^\times)$ and $(\mathbb{R}^n, \mathbb{Q}^n \rtimes \Gamma)$. Now Theorem 8.1 yields the required result. This completes the proof. \square

Example 2: Let A be an $n \times n$ matrix with integer entries such that $\det(A) \neq 0$ and $\bigcap_{r=0}^\infty A^r \mathbb{Z}^n = \{0\}$. Let $\Gamma := \{A^r : r \in \mathbb{Z}\} \cong \mathbb{Z}$. Denote the subgroup N_Γ by N_A and the Cuntz-Li algebra $\mathfrak{A}[N_\Gamma \rtimes \Gamma, \mathbb{Z}^n]$ by \mathfrak{A}_A . Denote the transpose A^t by B . Then $\Gamma^{op} = \{B^r : r \in \mathbb{Z}\} \cong \mathbb{Z}$.

We claim that the duality result is applicable to this example. The only thing that needs verification is $\bigcap_{r=0}^\infty B^r \mathbb{Z}^n = \{0\}$. This follows from the following lemma.

Lemma 8.8. *Let A be a $n \times n$ matrix with integer entries and denote A^t by B . Then $\bigcap_{r=0}^\infty A^r \mathbb{Z}^n = \{0\}$ if and only if $\bigcap_{r=0}^\infty B^r \mathbb{Z}^n = \{0\}$.*

Proof. Since A and B are similar over \mathbb{Q} , it follows that there exists $Y \in GL_n(\mathbb{Q})$ such that $YAY^{-1} = B$. Choose a non-zero integer m such that $X = mY \in M_n(\mathbb{Z})$. One has $XA = BX$. By induction, it follows that $XA^r = B^rX$ for every $r \geq 0$. First note that it is enough to show that $\bigcap_{r=0}^\infty A^r \mathbb{Z}^n \neq \{0\}$ implies $\bigcap_{r=0}^\infty B^r \mathbb{Z}^n \neq \{0\}$.

Suppose v is a non-zero element in $\bigcap_{r=0}^{\infty} A^r \mathbb{Z}^n$. Then

$$\begin{aligned} Xv &\in \bigcap_{r=0}^{\infty} XA^r \mathbb{Z}^n \\ &= \bigcap_{r=0}^{\infty} B^r X \mathbb{Z}^n \subset \bigcap_{r=0}^{\infty} B^r \mathbb{Z}^n. \end{aligned}$$

Since X is invertible over \mathbb{Q} , it follows that Xv is a non-zero element in $\bigcap_{r=0}^{\infty} B^r \mathbb{Z}^n$. Thus if $\bigcap_{r=0}^{\infty} A^r \mathbb{Z}^n \neq \{0\}$ then $\bigcap_{r=0}^{\infty} B^r \mathbb{Z}^n \neq \{0\}$. This completes the proof. \square

Now Theorem 8.1 and Proposition 8.5 implies the following proposition.

Proposition 8.9. *The C^* -algebra \mathfrak{A}_{A^t} is Morita-equivalent to $C_0(\mathbb{R}^n) \rtimes (N_A \rtimes \mathbb{Z})$. Also \mathfrak{A}_{A^t} is Morita-equivalent to $(C^*(N_A) \rtimes \mathbb{R}^n) \rtimes \mathbb{Z}$.*

Proposition 8.9 for the case when $n = 1$ and $A = (2)$ was proved in [SL10a]. In this case, the C^* -algebra $\mathfrak{A}_{A^t} = \mathfrak{A}_A$ is the C^* -algebra \mathcal{Q}_2 considered in [SL10a]. The subgroup $\bigcup_{r=0}^{\infty} 2^{-r} \mathbb{Z}$ is denoted $\mathbb{Z}[\frac{1}{2}]$ in [SL10a]. The Morita equivalence between \mathcal{Q}_2 and $C_0(\mathbb{R}) \rtimes (\mathbb{Z}[\frac{1}{2}] \rtimes (2))$ is called the 2-adic duality theorem in [SL10a]. (Cf. Corollary 5.5 and Theorem 7.5 in [SL10a].)

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